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HARMONIC FUNCTIONS ON THE REAL HYPERBOLIC BALL II HARDY AND LIPSCHITZ SPACES

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ABSTRACT. **In this paper, we pursue the study of harmonic functions on the real hyperbolic ball started in [12]. Our focus here is on the theory of Hardy, Hardy-Sobolev and Lipschitz spaces of these functions. We prove here that these spaces admit Fefferman-Stein like characterizations in terms of maximal and square functionals. We further prove that the hyperbolic harmonic extension of Lipschitz functions on the boundary extend into Lipschitz functions on the whole ball.**

1. Introduction

In this article, the sequel of [12], we study Hardy, Hardy-Sobolev and Lipschitz spaces of harmonic functions on the real hyperbolic ball. There are two main motivations for doing so :

While studying Hardy spaces of Euclidean harmonic functions on the unit ball $\mathbb{B}_n$ of $\mathbb{R}^n$, one is often lead to consider estimates of these functions on balls with radius smaller than the distance of the center of that ball to the boundary $\mathbb{S}^{n-1}$ of $\mathbb{B}_n$. Thus hyperbolic geometry is implicitly used in the study of Euclidean harmonic functions.

The second motivation of this paper lies in the recent developments of the theory of Hardy and Hardy-Sobolev spaces of $\mathcal{M}$-harmonic functions related to the complex hyperbolic metric on the unit ball, as exposed in [1] and [2]. Our aim here is to develop a similar theory in the case of the real hyperbolic ball. In this paper, n will be an integer, $n \geq 3$ and p a real number, $0 < p < +\infty$.

Our starting point is a result of [12] stating that the Hardy spaces $\mathcal{H}^p$ of hyperbolic harmonic functions ($\mathcal{H}$-harmonic functions in the terminology of [12]) admit an atomic decomposition similar to the Euclidean harmonic functions. Then, for $0 < p < +\infty$, define the space $H^p(\mathbb{S}^{n-1})$ as $L^p(\mathbb{S}^{n-1})$ if $p > 1$ and as the equivalent of Garnett-Latter's atomic H^p-space if $0 < p \leq 1$ (*see* [12] for the exact definition). This space has been characterized in terms of square functionals of the Euclidean harmonic extensions of its elements by Colzani [4]. We will here give these Fefferman-Stein characterizations directly in terms of their $\mathcal{H}$-harmonic extensions. More precisely, for an $\mathcal{H}$-harmonic function u, we prove the expected equivalence between $u \in \mathcal{H}^p$ and its non-tangential maximal function, area integral or Littlewood-Paley g-function belonging to $L^p(\mathbb{S}^{n-1})$.

In doing so, a choice of two methods is presented to us. We may either use the link between $\mathcal{H}$-harmonic functions and Euclidean harmonic functions from [12] as for the atomic decomposition or else, adapt the proofs in Fefferman-Stein [8] to our context. In both cases some difficulties appear.

For the first method, the link we use only allows to transfer results from the interior of the hyperbolic ball to the interior of the Euclidean ball, and from there to the boundary $\mathbb{S}^{n-1}$ (by usual methods). Unfortunately a converse link that would allow us to go back from the Euclidean ball to the hyperbolic ball is only available in even dimension. Note also that another link back from the Euclidean ball to the hyperbolic ball has been exhibited in [15] –see [12], lemma 9– but this link implies loss of regularity and is thus not adapted to this context.

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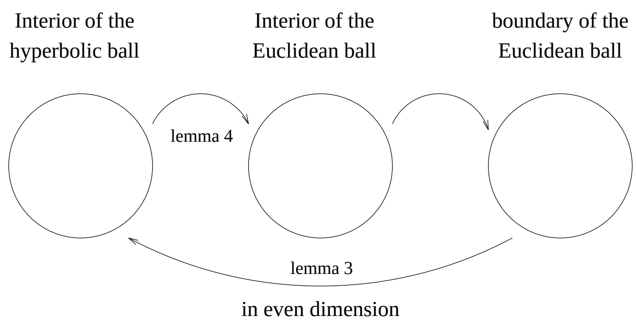

FIGURE 1. Links between hyperbolic and Euclidean harmonic functions

In order to present unified proofs independent of the parity of the dimension of the balls, we have thus inclined for Fefferman-Stein's method. In doing so, the main difficulty is that the hyperbolic Poisson kernels do not form a semi-group under convolution. In particular, if the function u is $\mathcal{H}$-harmonic, the function $u_\delta : x \mapsto u(\delta x)$ may not be $\mathcal{H}$-harmonic anymore. This leads us to introduce the concept of $\mathcal{H}_\delta$-harmonic functions and to get estimates on these functions.

Our next interest is in developing a theory of Hardy-Sobolev spaces of $\mathcal{H}$-harmonic functions, similar to the one developed in [2]. The first step is to prove mean value inequalities for $\mathcal{H}$-harmonic functions and their derivatives. This is done by adapting the proof in [2] using the theory of hypo-elliptic operators. We think that our mean value inequalities have an interest in their own and that the proof should adapt to all rank one spaces of the non-compact type. The remaining of the proofs are direct adaptations of [2]. However, as in [12] where it is proved that the boundary behavior of derivatives of $\mathcal{H}$-harmonic functions dependeds on the parity of the dimension of $\mathbb{B}_n$, it is proved here that the characterizations of Hardy-Sobolev spaces depend on the parity of the order of derivation. Note that Graham [9] has already notice a dependance of the behavior of harmonic functions on the parity of the dimension of the balls.

Finally, we take advantage of the link between Euclidean and hyperbolic harmonic functions to show how results on Lipschitz spaces of Euclidean harmonic functions (see [10]) can be transfered to the hyperbolic harmonic context. In particular, we show that the $\mathcal{H}$-harmonic extension of a Lipschitz function on the boundary is still a Lipschitz function of the same order on the whole ball. Further, we prove that in odd dimension, the limit-class preserved by $\mathcal{H}$-harmonic Poisson integrals is the Zygmund class of order n. This completes a result in [12] that states that this regularity is optimal in the sense that the $\mathcal{H}$-harmonic extension of a function on $\mathbb{S}^{n-1}$ is at most in this class.

This article is organized as follows. In the next section we present the setting of our problem and state our main results. Section 3 is devoted to the proofs of the technical lemmas we will need, including the mean value inequalities. In section 4 we prove the Fefferman-Stein characterization of our $\mathcal{H}^p$ spaces. The following section is devoted to the proofs of similar characterizations for Hardy-Sobolev spaces while in the last section we give the results on Lipschitz spaces.

2. Statement of the problem and results

2.1. $SO(n,1)$ **and its action on** $\mathbb{B}_n$**.** Let $G = SO(n,1) \subset GL_{n+1}(\mathbb{R})$, $(n \geq 3)$ be the identity component of the group of matrices $g = (g_{ij})_{0 \leq i,j \leq n}$ such that $g_{00} \geq 1$, $\det g = 1$ and that leave invariant the quadratic form $-x_0^2 + x_1^2 + \ldots + x_n^2$. G admits a Cartan decomposition $G = K\overline{A_+}K$ where

$$K = \left\{ k = \begin{pmatrix} 1 & 0 \\ 0 & \hat{k} \end{pmatrix} \ : \ \hat{k} \in SO(n) \right\}$$

and

$$A_+ = \left\{ a_t = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t & 0 \\ \operatorname{sh} t & \operatorname{ch} t & 0 \\ 0 & 0 & Id_{n-1} \end{pmatrix} \;:\; t \in \mathbb{R}_+ \right\}.$$

In this decomposition, every $g \in G$ can be written $g = k_g a_{t(g)} k_g^{'}$.

Let $|.|$ be the Euclidean norm on $\mathbb{R}^n$ and $\langle .,. \rangle$ the associated scalar product. Let $\mathbb{B}_n = \{x \in \mathbb{R}^n \;:\; |x| < 1\}$ and $\mathbb{S}^{n-1} = \partial\mathbb{B}_n = \{x \in \mathbb{R}^n \;:\; |x| = 1\}$. The homogeneous space G/K can be identified with $\mathbb{B}_n$, and it is well known (*see* [15]) that $SO(n,1)$ acts conformaly on $\mathbb{B}_n$ by $y = g.x$ with

$$y_p = \frac{\frac{1+|x|^2}{2} g_{p0} + \sum_{l=1}^n g_{pl} x_l}{\frac{1-|x|^2}{2} + \frac{1+|x|^2}{2} g_{00} + \sum_{l=1}^n g_{0l} x_l} \qquad \text{for } p = 1, \ldots, n.$$

The invariant measure on $\mathbb{B}_n$ is given by

$$d\mu = \frac{dx}{(1-|x|^2)^{n-1}} = \frac{r^{n-1} dr d\sigma}{(1-r^2)^{n-1}}$$

where dx is the Lebesgue measure on $\mathbb{B}_n$ and $d\sigma$ is the surface measure on $\mathbb{S}^{n-1}$.

We will need the following elementary facts about this action (see [11]):

Fact 1. *Let $g \in SO(n,1)$ and let $x_0 = g.0$. If $0 < \varepsilon < \frac{1}{6}$, then*

$$B\big(x_0, \frac{\sqrt{2}}{8}(1-|x_0|^2)\varepsilon\big) \subset g.B(0,\varepsilon) \subset B\big(x_0, 6(1-|x_0|^2)\varepsilon\big).$$

Fact 2. *Let $g \in SO(n,1)$ and let $x_0 = g.0$. Let v be a smooth function on $\mathbb{B}_n$ and define f on $\mathbb{B}_n$ by $f(x) = v(g.x)$. Then, for every k,*

$$(1-|x_0|^2)^k \big|\nabla^k v(x_0)\big| \leq C \big|\nabla^k f(0)\big|,$$

where $\left|\nabla^k\right|$ means $\sup\{\left|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\right| \;:\; |\alpha| \leq k\}$.

2.2. **The invariant laplacian on $\mathbb{B}_n$ and the associated Poisson kernel.** From [15] (see also [7],[6]), we know that the invariant laplacian on $\mathbb{B}_n$ for the considered action can be written as

$$D = (1-r^2)^2 \Delta + 2(n-2)(1-r^2) \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$$

where $r = |x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ and Δ is the Euclidean laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Note that D is given in radial-tangential coordinates by

$$D = \frac{1-r^2}{r^2} \big[(1-r^2) N^2 + (n-2)(1+r^2) N + (1-r^2) \Delta_\sigma\big]$$

with $N = r\frac{d}{dr} = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ and Δ_σ the tangential part of the Euclidean laplacian.

Definition. *A function u on $\mathbb{B}_n$ is $\mathcal{H}$-harmonic if $Du = 0$ on $\mathbb{B}_n$.*

Notation : Let $L = \frac{1}{r^2}\big[(1-r^2)N^2 + (n-2)(1+r^2)N + (1-r^2)\Delta_\sigma\big]$. Thus $Du = 0$ if and only if $Lu = 0$.

Green's formula for D is given by the following theorem :

Theorem 1 (Green's formula). *Let Ω be an open subset of $\mathbb{B}_n$ with $\mathcal{C}^1$ smooth boundary and let $\vec{n}$ be the exterior normal to $\partial\Omega$. Then for every functions $u, v \in \mathcal{C}^2(\Omega)$,*

$$\int_\Omega (1-|x|^2)^{-n} (uDv - vDu) dx = \int_{\partial\Omega} \left[u \frac{\partial v}{\partial \vec{n}} - v \frac{\partial u}{\partial \vec{n}}\right] (1-r^2)^{-n+2} d\sigma.$$

The Poisson kernel that solves the Dirichlet problem associated to D is given by

$$\mathbb{P}_h(r\eta,\xi)=\left(\frac{1-r^2}{1+r^2-2r\langle\eta,\xi\rangle}\right)^{n-1}$$

for $0\leq r<1$, $\eta,\xi\in\mathbb{S}^{n-1}$ *i.e.* for $r\eta\in\mathbb{B}_n$ and $\xi\in\mathbb{S}^{n-1}$.

Recall that the Euclidean Poisson kernel on the ball is given by

$$\mathbb{P}_e(r\eta,\xi)=\frac{1-r^2}{(1+r^2-2r\langle\eta,\xi\rangle)^{\frac{n}{2}}}$$

Notation : For a distribution φ on $\mathbb{S}^{n-1}$, we define $\mathbb{P}_e[\varphi]:\mathbb{B}_n\mapsto\mathbb{R}$ and $\mathbb{P}_h[\varphi]:\mathbb{B}_n\mapsto\mathbb{R}$ by

$$\begin{aligned}\mathbb{P}_e[\varphi](r\eta)=&\langle\varphi,\mathbb{P}_e(r\eta,.)\rangle\\ \mathbb{P}_h[\varphi](r\eta)=&\langle\varphi,\mathbb{P}_h(r\eta,.)\rangle\end{aligned}$$

$\mathbb{P}_e[\varphi]$ is the *Poisson integral* of φ, and $\mathbb{P}_h[\varphi]$ will be called the *$\mathcal{H}$-Poisson integral* of φ.

2.3. **Expansion of $\mathcal{H}$-harmonic functions in spherical harmonics.** Let ${}_2F_1$ denote Gauss' *hypergeometric function* and let $F_l(x)={}_2F_1(l,1-\frac{n}{2},l+\frac{n}{2};x)$ and $f_l(x)=\frac{F_l(x)}{F_l(1)}$. (See [5] for properties of ${}_2F_1$ used here).

In [13], [14] and [15], the spherical harmonic expansion of $\mathcal{H}$-harmonic functions has been obtained. Another proof, based on the method developped in [1] for $\mathcal{M}$-harmonic functions, can be found in [11]. We have the following :

Theorem 2. *Let u be an $\mathcal{H}$-harmonic function of class $\mathcal{C}^2$ on $\mathbb{B}_n$. Then the spherical harmonic expansion of u is given by*

$$u(r\zeta)=\sum_l f_l(r^2)u_l(r\zeta),$$

where this series is absolutely convergent and uniformaly convergent on every compact subset of $\mathbb{B}_n$.

Moreover, if we denote by $\mathbb{Z}_l^\zeta$ the zonal function of order l with pole ζ, then the hyperbolic Poisson kernel is given by

$$\mathbb{P}_h(r\zeta,\xi)=\sum_{l\geq0}\frac{F_l(r^2)}{F_l(1)}r^l\mathbb{Z}_l^\zeta(\xi).$$

Recall also that the Euclidean Poisson kernel is given by

$$\mathbb{P}_e(r\zeta,\xi)=\sum_{l\geq0}r^l\mathbb{Z}_l^\zeta(\xi).$$

In case the dimension n is *even*, this two kernels are linked by the following.

Lemma 3. *Assume n is even, and write $n=2p$. There exists p polynomials $P_0,P_1,\ldots,P_{p-1}$ such that, for every $r\zeta\in\mathbb{B}_n,\xi\in\mathbb{S}^{n-1}$,*

$$\mathbb{P}_h(r\zeta,\xi)=\sum_{k=0}^{p-1}P_k(r)(1-r^2)^k\frac{\partial^k}{\partial r^k}\mathbb{P}_e(r\zeta,\xi).$$

Proof. For $a\in\mathbb{R}$, write $(a)_k=\frac{\Gamma(a+k)}{\Gamma(a)}$. From [5] we get

$$F_l(x)={}_2F_1(l,1-p,l+p,x)=\frac{1}{(l+p)_{p-1}}\frac{(1-x)^{2p-1}}{x^{l+p-1}}\frac{d^{p-1}}{dx^{p-1}}\big(x^{l+2(p-1)}(1-x)^{-p}\big).$$

Let $\alpha_{l,j}$ be defined by $\alpha_l,0=1$ and $\alpha_{l,j+1}=\big(l+2(p-1)-j\big)\alpha_{l,j}$, then by Leibniz' formula

$${}_2F_1(l,1-p,l+p,x)=\frac{1}{(l+p)_{p-1}}\sum_{j=0}^{p-1}\binom{p-1}{j}(p)_j\alpha_{l,j}x^j(1-x)^j.$$

In particular ${}_2F_1(l,1-p,l+p,1)=\frac{1}{(l+p)_{p-1}}$ thus

$$\frac{F_l(x)}{F_l(1)}=\sum_{j=0}^{p-1}\binom{p-1}{j}(p)_j\alpha_{l,j}x^j(1-x)^j.$$

Furthermore, it is easy to see that one can write

$$\alpha_{l,j}=\sum_{k=0}^{j}a_{k,j}l(l-1)\dots(l-k+1)$$

where the coefficients $a_{k,j}$ are independent from l. It results from this and the spherical harmonics expansions of $\mathbb{P}_h$ and $\mathbb{P}_e$ that there exist polynomials $P_0,P_1,\dots,P_{p-1}$ such that, for every $r\zeta\in\mathbb{B}_n,\xi\in\mathbb{S}^{n-1}$,

$$\mathbb{P}_h(r\zeta,\xi)=\sum_{k=0}^{p-1}P_k(r)(1-r^2)^k\frac{\partial^k}{\partial r^k}\mathbb{P}_e(r\zeta,\xi).$$

which completes the proof. □

In [12], the following link between euclidean harmonic functions and $\mathcal{H}$-harmonic functions has been exhibited :

Lemma 4. *There exists a function $\eta:[0,1]\times[0,1]\mapsto\mathbb{R}^+$ such that*

i: $\mathbb{P}_e(r\zeta,\xi)=\int_0^1\eta(r,\rho)\mathbb{P}_h(\rho r\zeta,\xi)d\rho$,

ii: *for every k, there exists a constant C_k such that for every $r\in[0,1]$,*

$$\int_0^1\left|\left(r\frac{\partial}{\partial r}\right)^k\eta(r,\rho)\right|d\rho\leq\frac{C}{(1-r)^k}.$$

Proof. According to [12], the function η is given by

$$\eta(r,s)=c(1-r^2)(1-r^2s^2)^{2-n}\left[(1-s)(1-sr^2)\right]^{\frac{n}{2}-2}s^{\frac{n}{2}-1}.$$

The etimate *ii/* is easily obtained by differentiation. □

2.4. **Hardy and Hardy-Sobolev spaces.** The aim of this article is to extend Fefferman-Stein [8] theory to Hardy and Hardy-Sobolev spaces of $\mathcal{H}$-harmonic functions. We will therefore need to define analogs of non-tangential maximal functions, area integrals and Littlewood-Paley g functions.

Definition. *For $0<\alpha<1$ and $\zeta\in\mathbb{S}^{n-1}$, let $\mathcal{A}_\alpha(\zeta)$ be the interior of the convex hull of $B(0,\alpha)$ and ζ ; $\mathcal{A}_\alpha(\zeta)$ will be called non-tangential approach region.*

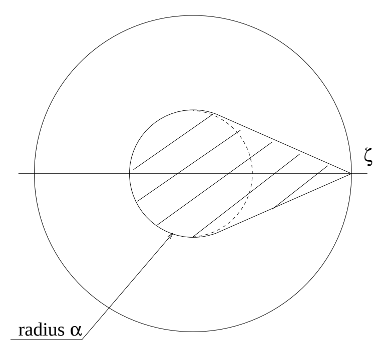

FIGURE 2. non-tangential approach region $\mathcal{A}_\alpha(\zeta)$

For a function u defined on $\mathbb{B}_n$, define the following functions on $\mathbb{S}^{n-1}$:

1. $\mathcal{M}[u](\xi)=\sup_{0<r<1}|u(r\xi)|$,

2. $\mathcal{M}_\alpha[u](\xi) = \sup_{x\in\mathcal{A}_\alpha(\xi)} |u(x)|$.
3. $S_\alpha[u](\xi) = \left[\int_{\mathcal{A}_\alpha(\xi)} |\nabla u(x)|^2 (1-|x|^2)^{-n+2} dx\right]^{\frac{1}{2}}$.
4. $S_\alpha^N[u](\xi) = \left[\int_{\mathcal{A}_\alpha(\xi)} |Nu(x)|^2 (1-|x|^2)^{-n+2} dx\right]^{\frac{1}{2}}$.
5. $g[u](\xi) = \left[\int_0^1 |\nabla u(t\xi)|(1-t^2)dt\right]^{\frac{1}{2}}$.
6. $g^N[u](\xi) = \left[\int_0^1 |Nu(t\xi)|(1-t^2)dt\right]^{\frac{1}{2}}$.

We can then define the Hardy spaces for $0 < p < +\infty$ as

$$\mathcal{H}^p = \{u\ \mathcal{H}-\text{harmonic} \ :\ \mathcal{M}[u] \in L^p(\mathbb{S}^{n-1})\}.$$

We will prove the following result :

Theorem A. *For $0 < p < 2$ and u $\mathcal{H}$-harmonic, the following are equivalent :*

1. $u \in \mathcal{H}^p$.
2. *u has a boundary distribution in $H^p(\mathbb{S}^{n-1})$.*
3. *$\mathcal{M}_\alpha[u] \in L^p(\mathbb{S}^{n-1})$ for some $0 < \alpha < 1$.*
4. *$S_\alpha[u] \in L^p(\mathbb{S}^{n-1})$ for some $0 < \alpha < 1$.*
5. *$S_\alpha^N[u] \in L^p(\mathbb{S}^{n-1})$ for some $0 < \alpha < 1$.*
6. $g[u] \in L^p(\mathbb{S}^{n-1})$,
7. $g^N[u] \in L^p(\mathbb{S}^{n-1})$.

Moreover, the equivalence of 1, 2 and 3 is valid for $0 < p < +\infty$.

Remark : This theorem implies, in particular, that if assertions 3, 4 and 5 are satisfied for some α, they are satified for every α.

Note that with lemma 3, part of this theorem is obvious in case the dimension n is even. However, we prefer giving here unified proofs independent of the parity of the dimension.

Define now the Hardy-Sobolev spaces for $0 < p < +\infty$ and $k \in \mathbb{N}$ as

$$\mathcal{H}_k^p = \{u\ \mathcal{H}-\text{harmonic} \ :\ \text{for all } j \le k,\ \mathcal{M}\left[\nabla^j u\right] \in L^p(\mathbb{S}^{n-1})\}.$$

and

$$H_k^p(\mathbb{S}^{n-1}) = \{f \in H^p(\mathbb{S}^{n-1})\ ;\ \nabla^j f \in H^p(\mathbb{S}^{n-1}),\ 0 \le j \le k\}.$$

We prove the following theorem :

Theorem B. *For $0 < p < 2$, for every integer $0 \le k \le n-2$ and for every $\mathcal{H}$-harmonic function u, the following are equivalent :*

1. $u \in \mathcal{H}_k^p$.
2. *u has a boundary distribution in $H_k^p(\mathbb{S}^{n-1})$.*
3. *u has a boundary distribution f satisfying $(-\Delta_\sigma)^{\frac{l}{2}} f \in H^p(\mathbb{S}^{n-1})$ for $0 \le l \le k$.*
4. *$u \in \mathcal{H}_{k-1}^p$ and for some α such that $0 < \alpha < 1$, $\mathcal{M}_\alpha\left[(-\Delta_\sigma)^{k/2} u\right] \in L^p(\mathbb{S}^{n-1})$.*
5. *$u \in \mathcal{H}_{k-1}^p$ and for some α such that $0 < \alpha < 1$, $S_\alpha\left[(-\Delta_\sigma)^{k/2} u\right] \in L^p(\mathbb{S}^{n-1})$.*
6. *$u \in \mathcal{H}_{k-1}^p$ and for some α such that $0 < \alpha < 1$, $S_\alpha^N\left[(-\Delta_\sigma)^{k/2} u\right] \in L^p(\mathbb{S}^{n-1})$.*
7. *$u \in \mathcal{H}_{k-1}^p$ and for some α such that $0 < \alpha < 1$, $S_\alpha\left[N^k u\right] \in L^p(\mathbb{S}^{n-1})$.*
8. *$u \in \mathcal{H}_{k-1}^p$ and for some α such that $0 < \alpha < 1$, $S_\alpha^N\left[N^k u\right] \in L^p(\mathbb{S}^{n-1})$.*
9. *$u \in \mathcal{H}_{k-1}^p$ and for some α such that $0 < \alpha < 1$, $S_\alpha\left[\nabla^k u\right] \in L^p(\mathbb{S}^{n-1})$.*

Moreover, the equivalence of 1, 2, 3 and 4 is valid for $0 < p < +\infty$.

Remark 1 : Again, this theorem implies that if assertions 4 to 9 are satisfied for some α, they are satified for every α.

Remark 2 : As $(-\Delta_\sigma)^{1/2}$ preserves $\mathcal{H}$-harmonicity, the equivalence of 2, 4, 5 and 6 means that $(-\Delta_\sigma)^{1/2}u \in \mathcal{H}^p$. The equivalence between 2 and 3 then follows from the atomic decomposition of $H^p(\mathbb{S}^{n-1})$ and standard singular integral arguments.

Remark 3 : Let $\mathcal{L}_{i,j} = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}$. They satisfy $\Delta_\sigma = \sum_{i<j}\mathcal{L}_{i,j}^2$. It is obvious that the $\mathcal{L}_{i,j}$'s commute with the invariant laplacian D so that they preserve $\mathcal{H}$-harmonicity. Further, if l is an odd integer with $l = 2l_0+1$, we can replace $(-\Delta_\sigma)^{l/2}$ in 4, 5 and 6 by the set of operators $\{\Delta_\sigma^{l_0}\mathcal{L}_{i,j}u \;:\; 0 \le i,j \le n\}$.

Define now

$$\mathcal{H}^p_{k,N} = \{u \in \mathcal{H}^p;\ \mathcal{M}\big[N^l u\big] \in L^p(\mathbb{S}^{n-1}),\ 0 \le l \le k\}.$$

We study the relationship between $\mathcal{H}^p_k$ and $\mathcal{H}^p_{k,N}$. The situation is slightly different as parity of the order of derivation is involved.

Theorem C. *For $0<\alpha<1$, $0<p<+\infty$, and k an integer, $0 \le k \le n-2$. Then*

1. *If k is* even, *the following are equivalent :*
   (a) $u \in \mathcal{H}^p_{k,N}$.
   (b) *For some α such that $0<\alpha<1$, for every $0 \le l \le k$, $\mathcal{M}_\alpha\big[N^l u\big] \in L^p(\mathbb{S}^{n-1})$.*
   (c) *$u \in \mathcal{H}^p_k$ and hence all the equivalent properties stated in theorem B are valid.*
2. *If k is* odd, *the following are equivalent :*
   (a) $u \in \mathcal{H}^p_{k,N}$
   (b) *For some α such that $0<\alpha<1$, for every $0 \le l \le k$, $\mathcal{M}_\alpha\big[N^l u\big] \in L^p(\mathbb{S}^{n-1})$.*
   (c) *$u \in \mathcal{H}^p_{k-1}$ and $\mathcal{M}\Big[(1-r^2)\Delta_\sigma^{\frac{k+1}{2}}u\Big] \in L^p(\mathbb{S}^{n-1})$.*

Remark : The third assertion in part 2, *i.e.* when k is odd, is in particular satisfied when $u \in \mathcal{H}^p_{k-1}$ and $\mathcal{M}_\alpha\Big[(-\Delta_\sigma)^{\frac{k}{2}}u\Big] \in L^p(\mathbb{S}^{n-1})$, that is when u is in $\mathcal{H}^p_k$. This gives the inclusion $\mathcal{H}^p_k \subset \mathcal{H}^p_{k,N}$. However, the space $\mathcal{H}^p_{k,N}$ is strictly bigger.

This result looks, at first sight, quite surprising since it is usually expected that the radial derivative dominates the gradient. In fact, this is naturally true in the interior of the domains and for instance, $S_\alpha(N^l u) \in L^p(\mathbb{S}^{n-1})$, $0 \le l \le k$ implies (and in fact is equivalent to) $u \in \mathcal{H}^p_k$ as stated in theorem B.

It is no longer true for conditions involving the behavior of the radial derivatives near the boundary. For instance, when $k=1$, we know from [12] that Nu has a boundary distribution that is identically zero. So, for u $\mathcal{H}$-harmonic, to be in $\mathcal{H}^p_{1,N}$ can not be translated as a constraint on the boundary behaviour of u.

3. Preliminary lemmas

3.1. **Mean value inequalities.** Recall that $\mathcal{H}$-harmonic functions satisfy the following mean value equalities :

Let $a \in \mathbb{B}_n$ and $g \in SO(n,1)$ such that $g.0 = a$. Then, for every $\mathcal{H}$-harmonic function u,

$$u(a) = \frac{1}{\mu\big(B(0,r)\big)}\int_{g.B(0,r)} u(x)\,d\mu(x).$$

Thus, with fact 1 and $d\mu = \frac{dx}{1-|x|^2}$, we get

$$|u(a)| \le \frac{C}{(1-|a|^2)^n}\int_{B\left(a,6(1-|a|^2)\varepsilon\right)} |u(x)|\,dx \tag{3.1}$$

We will also need mean value inequalities for normal derivatives of $\mathcal{H}$-harmonic functions, in particular when we study Hardy-Sobolev spaces. But, normal derivatives of $\mathcal{H}$-harmonic functions are no longer $\mathcal{H}$-harmonic, so that inequality (3.1) does not apply to them.

To obtain this inequalities, we will follow the main lines of the proof in [2] for $\mathcal{M}$-harmonic functions.

Therefore, we will first study the commutator between N^k and L (which is easier to compute than the commutator between N^k and D). This leads us to the existence of an elliptic operator $\mathbb{N}_q$ such

that for every $\mathcal{H}$-harmonic function u, $N^k u$ is annihilated by $\mathbb{N}_q$. We can then apply L^2 theory of elliptic operators and get estimates for $N^k u$ in 0. To obtain the estimates in an arbitrary point a of $\mathbb{B}_n$, we transport the result from 0 to a with help of the action of $SO(n,1)$ on $\mathbb{B}_n$ by computing the action of $g \in SO(n,1)$ on $\mathbb{N}_q$.

Note that

$$LN - NL = 2L + 2(N^2 + \Delta_\sigma) - 2(n-2)N \tag{3.2}$$

Moreover an easy induction argument shows that there exist two sequences of polynomials $(P_k)_{k\geq 1}$ and $(Q_k)_{k\geq 1}$ of degree $k-1$ such that for $k \geq 1$,

$$LN^k = (N+2I)^k L + P_k(N)N^2 + Q_k(N)\Delta_\sigma - 2(n-2)(N+2I)^{k-1}N.$$

From this, using the same induction as in [2], we get

Proposition 5. *For every k, there exist polynomials $S_k(x,y)$ of degree at most $q-1$ (with $q = 2^{k-1}$) and $R_k(x,y) = x^q + \dots$ such that, if u is $\mathcal{H}$-harmonic, then*

$$L\big(R_k(L,\Delta_\sigma) - S_k(L,\Delta_\sigma)N\big)N^k u = 0.$$

We thus conclude that if u is $\mathcal{H}$-harmonic, then $v = N^k u$ is a solution of an equation $\mathbb{N}_q v = 0$ with $\mathbb{N}_q = L^{q+1} + \dots$ and $q = 2^{k-1}$.

We will use the following formalism : if $\mathbb{M}$ is a differential operator and Φ a diffeomorphism of $\mathbb{B}_n$ and if $f = v \circ \Phi$, then define $\Phi^*\mathbb{M}$ by

$$\Phi^*\mathbb{M}(f) = (\mathbb{M}v)\circ\Phi.$$

It is then obvious that

$$\Phi^*(\mathbb{M}_1 \circ M_2) = (\Phi^*\mathbb{M}_1)\circ(\Phi^*\mathbb{M}_2) \tag{3.3}$$

$$\Phi^*(h\mathbb{M}) = h(\Phi).\Phi^*\mathbb{M} \tag{3.4}$$

We will consider $v = N^k u$ with u $\mathcal{H}$-harmonic, so that $\mathbb{N}_q v = 0$ with $\mathbb{N}_q = L(R_k(L,\Delta_\sigma) - P_k(L,\Delta_\sigma)N) = L^{q+1} + \dots$

Let $g \in SO(n,1)$ be such that $g.0 = \rho\zeta = a \in \mathbb{B}_n$ and let $\Phi_a : \begin{array}{ccc} \mathbb{B}_n & \mapsto & \mathbb{B}_n \\ x & \mapsto & g.x \end{array}$.

But, by definition, D is invariant by the action of $SO(n,1)$ on $\mathbb{B}_n$, that is $\Phi_a{}^*D = D$. On the other hand, $D = (1-|x|^2)L$ thus (3.4) tells us that $\Phi_a{}^*D = (1-|\Phi_a(x)|^2)\Phi_a{}^*L$, which implies that $\Phi_a{}^*L = \frac{1-|x|^2}{1-|g.x|^2}L$, and the formula of [15] page 39 gives

$$\Phi_a{}^*L = \frac{(1+\rho^2|x|^2 - 2\rho\langle x,\zeta\rangle)^2}{1-\rho^2}L.$$

Further $\Phi_a{}^*N$ is a differential operator of order 1 with $\mathcal{C}^\infty$ coefficients defined by

$$\begin{aligned}\Phi_a{}^*Nf(x) =& \big\langle \Phi_a(x), dv_{\Phi_a(x)}\big\rangle = \big\langle \Phi_a(x), d(f\circ\Phi_a^{-1})_{\Phi_a(x)}\big\rangle \\ =& \big\langle \Phi_a(x), df_z.d(\Phi_a^{-1})_{\Phi_a(x)}\big\rangle\end{aligned}$$

thus, if $x \in B(0,\varepsilon)$ then, with fact 1, $\Phi_a(x) \in B\big(a, 6(1-a^2)\varepsilon\big)$, and with fact 2 (applied to $v(x) = x$), the coefficients of $(1-|a|^2)\Phi_a{}^*N$ as well as their derivatives are $\mathcal{C}^\infty$ and bounded independently of a.

As $\Phi_a{}^*N^2 = (\Phi_a{}^*N)\circ(\Phi_a{}^*N)$, $(1-|a|^2)^2\Phi_a{}^*N^2$ is a differential operator of order 2 with $\mathcal{C}^\infty$ coefficients bounded (as well as their derivatives) independently of a.

At last, $\Delta_\sigma = \frac{r^2}{1-r^2}L - N^2 - (n-2)\frac{1+r^2}{1-r^2}N$ thus $(1-|a|^2)\Phi_a{}^*\Delta_\sigma$ is also a differential operator of order 2 with $\mathcal{C}^\infty$ coefficients bounded (as well as their derivatives) independently of a.

Finally, $\mathbb{N}_q = L^{q+1}+$ terms of order $\leq 2q$ in L, Δ_σ and N with $\mathcal{C}^\infty$ coefficients, thus

$$\begin{aligned}
{\Phi_a}^*\mathbb{N}_q =&{\Phi_a}^* L^{q+1} + \text{terms of order } \leq 2q \text{ with } \mathcal{C}^\infty \text{ coefficients}\\
=&\frac{(1+\rho^2|x|^2 - 2\rho\langle x,\zeta\rangle)^{2(q+1)}}{(1-\rho^2)^{q+1}} L^{q+1}\\
&+ \text{terms of order } \leq 2q \text{ with } \mathcal{C}^\infty \text{ coefficients}\\
=&\frac{|\rho x-\zeta|^{2(q+1)}}{(1-\rho^2)^{q+1}} L^{q+1} + \mathbb{R}_{q,a}
\end{aligned}$$

where $\mathbb{R}_{q,a}$ is a differential operator of order $\leq 2q$ with $\mathcal{C}^\infty$ coefficients.

Let u be an $\mathcal{H}$-harmonic function, $v = N^k u$ and $f = v \circ \Phi_a$. As v satisfies $\mathbb{N}_q v = 0$, f satisfies $(1-|a|^2)^{q+1}{\Phi_a}^*\mathbb{N}_q f = 0$ on $B(0,\varepsilon)$ (with *e.g.* $\varepsilon < \frac{1}{6}$). We have thus shown that $(1-\rho^2)^{q+1}{\Phi_a}^*\mathbb{N}_q$ satisfies on $B(0,\varepsilon)$, $\varepsilon < 1/6$, all the hypotheses (with constants independent on a) of the following theorem (see [2] page 678):

Theorem 6. *Suppose $P(d)$ is a differential operator in $\mathbb{R}^N$,*

$$P(D) = \sum_{|\alpha|\leq 2q} h_\alpha(x) D^\alpha \quad \text{where} \quad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}},$$

which is elliptic with constant c_0 in $B(0,\varepsilon)$, that is,

$$\sum_{|\alpha|=2q} h_\alpha(x)\xi^\alpha \geq c_0|\xi|^{2q} \quad \text{for} \quad \xi \in \mathbb{R}^N,$$

and with $h_\alpha \in \mathcal{C}^\infty\left(\overline{B(0,\varepsilon)}\right)$. Assume that $P(D)f = 0$ in $B(0,\varepsilon)$. Then, for all non-negative integers m and all p such that $0 < p < \infty$,

$$|\nabla^m f(0)| \leq C \left(\int_{|x|\leq\varepsilon} |f(x)|^p dx\right)^{1/p},$$

where C depends only on c_0, ε, m, p and a bound of the norms of the functions h_α in some $\mathcal{C}^l\left(\overline{B(0,\varepsilon)}\right)$-space with $l = l(m)$

From this, we get

$$\left|\nabla^d f(0)\right| \leq c\left(\int_{|x|\leq\varepsilon} |f(x)|^p dx\right)^{\frac{1}{p}} \leq c\left(\int_{|x|\leq\varepsilon} |f(x)|^p \frac{dx}{(1-|x|^2)^n}\right)^{\frac{1}{p}}$$

or, with *Fact 2*,

$$\left|\nabla^d v(a)\right| \leq c\left(\int_{B(0,\varepsilon)} |v\circ\Phi_a(x)|^p d\mu(x)\right)^{\frac{1}{p}} \times (1-|a|^2)^{-d}$$

where μ is the G-invariant measure on $\mathbb{B}_n$. Thus

$$\left|\nabla^d v(a)\right| \leq c\left(\int_{g.B(0,\varepsilon)} |v(x)|^p d\mu(x)\right)^{\frac{1}{p}} \times (1-|a|^2)^{-d}$$

and, with *Fact 1*,

$$\begin{aligned}\left|\nabla^d v(a)\right| \leq & c\left(\int_{B\left(a,6(1-|a|^2)\varepsilon\right)} |v(x)|^p \frac{dx}{(1-|x|^2)^n}\right)^{\frac{1}{p}} \times (1-|a|^2)^{-d}\\ \leq & c(1-|a|^2)^{-d-\frac{n}{p}}\left(\int_{B\left(a,6(1-|a|^2)\varepsilon\right)} |v(x)|^p dx\right)^{\frac{1}{p}}\end{aligned}$$

In conclusion, we have just proved the following lemma

Lemma 7 (Mean Value Inequality). *For every $0<\varepsilon<\frac{1}{6}$, $k,d\in\mathbb{N}$, $0<p<+\infty$, there exists a constant c such that, for every $\mathcal{H}$-harmonic function u, and every $a\in\mathbb{B}_n$,*

$$\left|\nabla^d N^k u(a)\right| \leq c(1-|a|)^{-d-\frac{n}{p}}\left(\int_{B\left(a,6(1-|a|^2)\varepsilon\right)} \left|N^k u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

Remark : In case $d=0$ ($\nabla^0=I$), $k=0$ and $p=1$, we again obtain inequality (3.1).

Let $\mathcal{L}_{i,j}=x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i}$ ($1\leq i<j\leq n$). One easily sees that $N\mathcal{L}_{i,j}=\mathcal{L}_{i,j}N$ and that $\Delta\mathcal{L}_{i,j}=\mathcal{L}_{i,j}\Delta$, thus $L\mathcal{L}_{i,j}=\mathcal{L}_{i,j}L$ and $D\mathcal{L}_{i,j}=\mathcal{L}_{i,j}D$. In particular, if u is $\mathcal{H}$-harmonic, so is $\mathcal{L}_{i,j}^k u$ for every $k\in\mathbb{N}$. Applying lemma 7 to $\mathcal{L}_{i,j}u$, for every $0<\varepsilon<\frac{1}{6}$ and every $0<p<+\infty$, there exists a constant C such that for every $\mathcal{H}$-harmonic function u, for every $1\leq i<j\leq n$ and every $k\in\mathbb{N}$, for every d, and every $a\in\mathbb{B}_n$,

$$\left|\nabla^d \mathcal{L}_{i,j}^k u(a)\right| \leq C(1-|a|)^{-d-\frac{n}{p}}\left(\int_{B\left(a,6(1-|a|^2)\varepsilon\right)} \left|\mathcal{L}_{i,j}^k u(x)\right|^p dx\right)^{\frac{1}{p}}. \tag{3.5}$$

Remark : Let $\tilde{\nabla}^k u$ be defined by

$$\{\mathbb{X}N^q u\ :\ \mathbb{X}=\prod_{l=1}^{p}\mathcal{L}_{i_l,j_l},\ p+q\leq k\},$$

then (3.5) implies that lemma 7 stays true if we replace N^k by $\tilde{\nabla}^k u$. But, outside a fixed neighborhood V of 0, $\left|\tilde{\nabla}^k u\right|\simeq\left|\nabla^k u\right|$, thus for every $a\in\mathbb{B}_n\setminus V$

$$\left|\nabla^d\nabla^k u(a)\right| \leq C(1-|a|)^{-d-\frac{n}{p}}\left(\int_{B\left(a,2(1-|a|^2)\varepsilon\right)} \left|\nabla^k u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

As for $a\in V$ one can apply theorem 6 on $B\big(a,2(1-|a|^2)\varepsilon\big)$ with constants independent of a, we get the previous inequality on V (recall that ∇^k means the set of all derivatves of order less than k). We thus get the following proposition :

Proposition 8. *For every $0<\varepsilon<\frac{1}{6}$ and every $0<p<+\infty$, there exists a constant C such that for every $\mathcal{H}$-harmonic function u, every $k\in\mathbb{N}$, $d\geq 0$, and for every $a\in\mathbb{B}_n$,*

$$\left|\nabla^{k+d}u(a)\right| \leq C(1-|a|)^{-d-\frac{n}{p}}\left(\int_{B\left(a,6(1-|a|^2)\varepsilon\right)} \left|\nabla^k u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

Remark 1 : In the sequel, we will not distinguish anymore between ∇^k and $\tilde{\nabla}^k$.

Remark 2 : The previous inequality can be restated in an invariant form under the action of the group, using invariant gradient and, more generally covariant derivation. if

Corollary 9. *Let $0<\alpha<\beta<1$, $k,d\in\mathbb{N}$. Then there exists a constant c such that for every $\mathcal{H}$-harmonic function u,*

$$\mathcal{M}_\alpha\big((1-|z|)^d\nabla^d N^k u\big)\leq c\mathcal{M}_\beta(N^k u).$$

Proof. This is an immediate consequence of lemma 7 and the fact that if $\alpha < \beta$ and if ε is small enough then, for every $\xi \in \mathbb{S}^{n-1}$ and every $a \in \mathcal{A}_\alpha(\xi)$, $B(a, 6(1-|a|^2)\varepsilon) \subset \mathcal{A}_\beta(\xi)$. □

3.2. **Integration over non-tangential approach regions.** For F a closed subset of $\mathbb{S}^{n-1}$, the tent over F is defined by

$$\mathcal{R}_\alpha(F) = \bigcup_{\xi \in F} \mathcal{A}_\alpha(\xi).$$

We will need the two following lemmas for integration over tents. Their proofs are similar to the ones for integration over tents in $\mathbb{R}_+^{n+1}$ (*see* [17]).

Lemma 10. *For every $0 < \alpha < 1$, there exists a constant C_α such that for every positive function Φ,*

$$\int_F \left\{ \int_{\mathcal{A}_\alpha(\xi)} \Phi(x)dx \right\} d\sigma(\xi) \le C_\alpha \int_{\mathcal{R}_\alpha(F)} \Phi(x)(1-|x|)^{n-1}dx.$$

As in the $\mathbb{R}_+^{n+1}$ case, the converse of this lemma is more complicated : let F be a closed subset of $\mathbb{S}^{n-1}$ and let $0 < \gamma < 1$. A point ξ of $\mathbb{S}^{n-1}$ is called a γ-*density* point of F if

$$\frac{\sigma[B(\xi) \cap F]}{\sigma[B(\xi) \cap \mathbb{S}^{n-1}]} \ge \gamma$$

for every ball $B(\xi)$ centered in ξ.

We will denote by F^* the set of γ-density points of F. The converse of lemma 10 is then :

Lemma 11. *Let $0 < \alpha < 1$. Then there exists γ, $0 < \gamma < 1$ sufficiently near to 1 such that, for every closed subset F of $\mathbb{S}^{n-1}$ and every positive function Φ, we have*

$$\int_{\mathcal{R}_\alpha(F^*)} \Phi(x)(1-|x|)^{n-1}dx \le C_{\alpha,\gamma} \int_F \left\{ \int_{\mathcal{A}_\alpha(\xi)} \Phi(x)dx \right\} d\sigma(\xi).$$

A direct consequence of these two lemmas is the following (see [3]) :

Lemma 12. *For $0 < p < 2$, for $0 < \alpha, \beta < 1$, there exists constants C_1, C_2 such that for every $\mathcal{C}^1$ function u on $\mathbb{B}_n$,*

$$C_1 \|S_\alpha[u]\|_{L^p(\mathbb{S}^{n-1})} \le \|S_\beta[u]\|_{L^p(\mathbb{S}^{n-1})} \le C_2 \|S_\alpha[u]\|_{L^p(\mathbb{S}^{n-1})}.$$

Similar estimates are valid if we replace S_α by S_α^N.

3.3. **Consequences of the mean value inequalities.** Let $l \in \mathbb{R}$ and f a function defined on $\mathbb{B}_n$. Define $I_l f$ by

$$I_l f(r\zeta) = \int_0^r f(t\zeta)(1-t)^{l-1}dt, \qquad 0 < r < 1,\ \zeta \in \mathbb{S}^{n-1}.$$

The following lemma is a direct consequence of the mean value inequalities and its proof follows the main lines of the upper half-line case (*see* [16] pages 214–216) or the $\mathcal{M}$-harmonic function case in [2].

Lemma 13. *For $0 < \alpha < \beta < 1$, $\gamma > -\frac{n}{2}$, $l \in \mathbb{R}$ and $d \in \mathbb{N}$, there exists a constant C such that, for every $\zeta \in \mathbb{S}^{n-1}$, and for every $\mathcal{H}$-harmonic function u*

$$\int_{\mathcal{A}_\alpha(\zeta)} \left[I_l(\nabla^d N^k u)\right](z)^2 (1-|z|)^{2\gamma} dz \le C \int_{\mathcal{A}_\beta(\zeta)} \left|N^k u(z)\right|^2 (1-|z|)^{2(l+\gamma-d)} dz.$$

Remark : If l is a positive integer, then $N^l h = g$ implies

$$|h| \le C\left[I_l|g| + \max_{j \le l-1, |z|<\varepsilon} \left|\nabla^j h(z)\right| \right].$$

This leads to the following lemma (*see* [2] for the proof in case of $\mathcal{M}$-harmonic functions) :

Lemma 14. *For $0<\alpha<\beta<1$, $\gamma>-\frac{n}{2}$ and $d\in\mathbb{N}$, there exists a constant C such that, for every $\zeta\in\mathbb{S}^{n-1}$, and every $\mathcal{H}$-harmonic function u,*

$$\int_{\mathcal{A}_\alpha(\zeta)}\left|\nabla^d u\right|(z)^2(1-|z|)^{2\gamma}dz\leq C\int_{\mathcal{A}_\beta(\zeta)}\left|N^k u(z)\right|^2(1-|z|)^{2(k+\gamma-d)}dz+C\sup_{|z|<\varepsilon}\left|\nabla^{k-1}u(z)\right|^2.$$

The last lemma we will need is also similar to the $\mathbb{R}^{n+1}_+$ case ([16], page 207) and results directly from the mean value inequality :

Lemma 15. *Let $0<\alpha<\beta<1$, There exists a constant C such that for every $\xi\in\mathbb{S}^{n-1}$ and for every $\mathcal{H}$-harmonic function u*

1. *if $|u|\leq 1$ on $\mathcal{A}_\beta(\xi)$ then $\left|(1-|x|^2)\nabla u\right|\leq C$ on $\mathcal{A}_\alpha(\xi)$,*
2. *if $S_\beta[u](\xi)\leq 1$ then $\left|(1-|x|^2)\nabla u\right|\leq C$ on $\mathcal{A}_\alpha(\xi)$.*

4. Characterization of $\mathcal{H}^p$ by maximal functions, area integrals and Littlewood-Paley g-functions

In this section we extend the theory of Fefferman-Stein [8] *to the $\mathcal{H}^p$ spaces.*

4.1. **Maximal Characterization of $\mathcal{H}^p$.** From the definition of $\mathcal{M}[u]$ and $\mathcal{M}_\alpha[u]$, it is obvious that $\mathcal{M}[u]\leq\mathcal{M}_\alpha[u]$, in particular, if $\mathcal{M}_\alpha[u]\in L^p(\mathbb{S}^{n-1})$ then $\mathcal{M}[u]\in L^p(\mathbb{S}^{n-1})$. The next proposition claims that the converse is true for $\mathcal{H}$-harmonic functions as well as for their normal derivatives.

Proposition 16. *For $0<\alpha<1$, $0<p<+\infty$, for every integer $k\geq 0$ and for every $\mathcal{H}$-harmonic function u, the following are equivalent :*

1. $\mathcal{M}\left[N^k u\right]\in L^p(\mathbb{S}^{n-1})$,
2. $\mathcal{M}_\alpha\left[N^k u\right]\in L^p(\mathbb{S}^{n-1})$.

Moreover, there exists $C=C_{\alpha,p}$ such that for every $\mathcal{H}$-harmonic function u,

$$\left\|\mathcal{M}\left[N^k u\right]\right\|_p\leq\left\|\mathcal{M}_\alpha\left[N^k u\right]\right\|_p\leq C\left\|\mathcal{M}\left[N^k u\right]\right\|_p.$$

Proof. According to lemma 7, for $a\in\mathcal{A}_\alpha(\zeta)$

$$\left|N^k u(a)\right|^{\frac{p}{2}}\leq C(1-|a|)^{-n}\int_{B\left(a,2(1-|a|)\varepsilon\right)}\left|N^k u(\omega)\right|^{\frac{p}{2}}d\omega.$$

Integrating in polar coordinates $\omega=r\eta$, we see that $\eta\in B\big(\zeta,c(1-|a|)\big)$ and bounding $\left|N^k u(\omega)\right|$ by $\mathcal{M}\left[N^k u\right](\zeta)$ we get

$$\begin{aligned}\left|N^k u(a)\right|^{\frac{p}{2}}\leq & C(1-|a|)^{-n}\int_{B\left(\zeta,c(1-|a|)\right)\cap\mathbb{S}^{n-1}}\left[\mathcal{M}\left[N^k u\right](\xi)\right]^{\frac{p}{2}}d\sigma(\xi)\times\int_{|a|-2(1-|a|)\varepsilon}^{|a|+2(1-|a|)\varepsilon}r^{n-1}dr\\ \leq & C(1-|a|)^{-n+1}\int_{B\left(\zeta,c(1-|a|)\right)\cap\mathbb{S}^{n-1}}\left[\mathcal{M}\left[N^k u\right](\xi)\right]^{\frac{p}{2}}d\sigma(\xi).\end{aligned}$$

But $\sigma\left[B\big(\zeta,c(1-|a|)\big)\cap\mathbb{S}^{n-1}\right]\sim(1-|a|)^{n-1}$ therefore

$$\mathcal{M}_\alpha\left[N^k u(\zeta)\right]^{\frac{p}{2}}\leq C\mathcal{M}_{HL}\left[\mathcal{M}\left[N^k u\right]^{\frac{p}{2}}\right](\zeta)$$

where $\mathcal{M}_{HL}$ is Hardy-Littlewood's maximal function on $\mathbb{S}^{n-1}$. We just have to use the fact that $\mathcal{M}_{HL}$ is bounded $L^2(\mathbb{S}^{n-1})\mapsto L^2(\mathbb{S}^{n-1})$ to complete the proof. □

Remark 1 : This proposition, whose proof is directly inspired from the $\mathbb{R}^{n+1}_+$ case in [8] depends only on the mean value inequalities (lemma 7). Thus, it remains true if we replace N^k by ∇^k or by $\mathcal{L}^k_{i,j}$ (thus also by $(-\Delta_\sigma)^{k/2}$) as long as we replace lemma 7 by proposition 8 or by inequality (3.5).

Remark 2 : For $k=0$ this is equivalence (1) ⇔ (2) of theorem A.

4.2. $\mathcal{H}_\delta$**-harmonic functions.** To take advantage of inequalities on harmonic functions on $\mathbb{R}^{n+1}_+$, one is often led to introduce the function $u_\varepsilon(x,t) = u(x, t+\varepsilon)$ which is still harmonic if u is, and which is smooth up to the boundary. One then hopes to get estimates that are independent of ε and then let ε go to 0.

In the case of $\mathcal{H}$-harmonic functions, we would like to operate in the same way. Unfortunately, if u is $\mathcal{H}$-harmonic, the function $u_\varepsilon(x) = u\big((1-\varepsilon)x\big)$ may not be $\mathcal{H}$-harmonic. We are thus led to introduce the following notion.

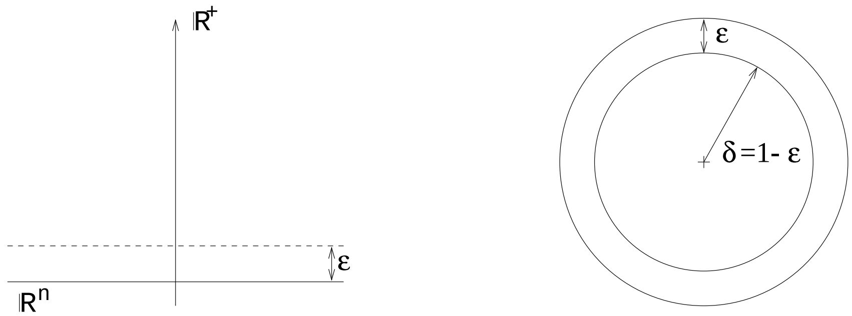

FIGURE 3. Function u_ε

Definition. *Let $0<\delta<1$ and let D_δ be the operator defined by*

$$D_\delta = (1-\delta^2 r^2)^2\Delta + 2(n-2)\delta^2(1-\delta^2 r^2)\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}.$$

We will say that u a smooth function on $\mathbb{B}_n$ is $\mathcal{H}_\delta$-harmonic if $D_\delta u = 0$.

An easy computation shows that if u is $\mathcal{H}$-harmonic, then the function v defined by $v(x) = u(\delta x)$ is $\mathcal{H}_\delta$-harmonic, *i.e.* $D_\delta u = 0$ or also $L_\delta u = 0$ with

$$L_\delta = (1-\delta^2 r^2)\Delta + 2(n-2)\delta^2 \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}.$$

To this laplacian, one can associate the $\mathcal{H}_\delta$-Poisson kernel given by its spherical harmonics expansion

$$\mathbb{P}_{h,\delta}(r\zeta,\xi) = \sum_{l=0}^\infty f_l(\delta^2 r^2) r^l \mathbb{Z}^l_\zeta(\xi),$$

to which we can associate $\mathcal{H}_\delta$-Poisson integrals.

Note also that we obtain a family of operators D_δ such that $D_0 = \Delta$, the Euclidean laplacian and $D_1 = D$ the hyperbolic laplacian. Similarely, notice that $\mathbb{P}_{h,0} = \mathbb{P}_e$ and $\mathbb{P}_{h,1} = \mathbb{P}_h$.

Green's formula for this Laplacian is

$$\int_\Omega \big(u(x)D_\delta v(x) - v(x) D_\delta u(x)\big)\frac{dx}{(1-\delta^2|x|^2)^n} = \int_{\partial\Omega}\left[u\frac{\partial v}{\partial \vec n} - v\frac{\partial u}{\partial \vec n}\right]\frac{d\sigma}{(1-\delta^2 r^2)^{n-2}}.$$

One can check that proofs of chapter 3.1 can be reproduced for $\mathcal{H}_\delta$-harmonic functions, in particular, the mean value inequality (lemma 7) remains valid with constants independent from δ. More precisely, we obtain :

Lemma 17. *For every $\varepsilon < \frac{1}{6}$, $k, d \in \mathbb{N}$, $0 < p < +\infty$, there exists a constant c such that, for every $\frac{1}{2} < \delta \leq 1$, every $\mathcal{H}_\delta$-harmonic function u and every $a \in \mathbb{B}_n$,*

$$\left|\nabla^d N^k u(a)\right| \leq c(1-|a|)^{-d-\frac{n}{p}}\left(\int_{B(a,6(1-|a|^2)\varepsilon)} \left|N^k u(x)\right|^p dx\right)^{\frac{1}{p}}.$$

We will need the following estimates :

Proposition 18. *There exists a constant C such that for every $0<\delta<1$, every $x\in\mathbb{B}_n$ and every $\xi\in\mathbb{S}^{n-1}$,*

$$0\leq\mathbb{P}_{h,\delta}(x,\xi)\leq C\mathbb{P}_e(x,\xi).$$

In particular, for every $0<\alpha<1$, there exists a constant C such that if $f\in L^2(\mathbb{S}^{n-1})$ and $u=\mathbb{P}_{h,\delta}[f]$ then

$$\|\mathcal{M}_\alpha[u]\|_{L^2(\mathbb{S}^{n-1})}\leq C\|f\|_{L^2(\mathbb{S}^{n-1})}.$$

Conversely, there exists $\eta>0$ and a constant $C>0$ such that for all $0<\delta<1$, for all $\xi\in\mathbb{S}^{n-1}$ and all $x\in\mathcal{A}_\eta(\xi)$,

$$\mathbb{P}_{h,\delta}(x,\xi)\geq\frac{C}{(1-|x|^2)^{n-1}}.$$

Proof. $\mathcal{H}_\delta$-harmonic functions satisfy the maximum principle, so the Poisson kernel $\mathbb{P}_{h,\delta}$ is positive.

Fix $\xi_0\in\mathbb{S}^{n-1}$ and let $u(x)=\mathbb{P}_{h,\delta}(x,\xi_0)$. With the mean value inequality (lemma 17)

$$0\leq u(x)\leq\frac{C}{(1-|x|^2)^n}\int_{B(x,1-|x|^2)}u(y)dy\leq\frac{C}{(1-|x|^2)^n}\int_{|x|-(1-|x|^2)}^{|x|+(1-|x|^2)}\int_{\mathbb{S}^{n-1}}u(r\zeta)d\sigma(\zeta)r^{n-1}dr$$

and as $\int_{\mathbb{S}^{n-1}}u(r\zeta)d\sigma(\zeta)=1$, we get

$$\mathbb{P}_{h,\delta}(x,\xi_0)\leq\frac{C}{(1-|x|^2)^{n-1}},$$

with C independent from ξ_0 and from δ. But $\mathbb{P}_e(x,\xi_0)\simeq\frac{C_\alpha}{(1-|x|^2)^{n-1}}$ in $\mathcal{A}_\alpha(\xi_0)$ thus $\mathbb{P}_h\leq C\mathbb{P}_e$ in the interior of $\mathcal{A}_\alpha(\xi_0)$.

On the other hand, $N\mathbb{P}_e(r\zeta,\xi_0)$ has same sign as

$$-2r\big[(1-r)^2+r(1-<\zeta,\xi_0>)\big]-n(1-r^2)\big(r-1+(1-<\zeta,\xi_0>)\big)$$

so it is negative in $\mathbb{B}_n\setminus\mathcal{A}_\alpha(\xi_0)$ for α big enough. This leads to $D_\delta\mathbb{P}_e<0$ on $\mathbb{B}_n\setminus\mathcal{A}_\alpha(\xi_0)$ and $C\mathbb{P}_e(x,\xi_0)-\mathbb{P}_h(x,\xi_0)>0$ on the boundary of $\mathbb{B}_n\setminus\mathcal{A}_\alpha(\xi_0)$ (with C independent from δ and from ξ_0), thus, by the maximum principle $\mathbb{P}_{h,\delta}\leq C\mathbb{P}_e$ on $\mathbb{B}_n\setminus\mathcal{A}_\alpha(\xi_0)$. ◇

For the other inequality, first notice that

$$\mathbb{P}_{h,\delta}(r\zeta,\xi)=\sum_{l\geq0}f_l(r^2)r^l\mathbb{Z}^l_\zeta(\xi),$$

and as $\mathbb{Z}^l_\xi(\xi)=1$, it turns out that

$$\mathbb{P}_{h,\delta}(r\xi,\xi)\geq\mathbb{P}_e(r\xi,\xi)\geq\frac{C_1}{(1-r^2)^{n-1}}.$$

But $\mathbb{P}_{h,\delta}$ is $\mathcal{H}_\delta$-harmonic and therefore satisfies mean value inequalities (lemma 17), *i.e.*

$$\begin{aligned}|\nabla\mathbb{P}_{h,\delta}(x,\xi)|\leq&\frac{C}{(1-|x|^2)^{n+1}}\int_{B\left(x,6(1-|x|^2)\varepsilon\right)}|\mathbb{P}_{h,\delta}(y,\xi)|dy\\ \leq&\frac{C}{(1-|x|^2)^{n+1}}\int_{B\left(x,6(1-|x|^2)\varepsilon\right)}|\mathbb{P}_e(y,\xi)|dy\\ \leq&\frac{C_2}{(1-|x|^2)^n}.\end{aligned}$$

Thus, by the fundamental theorem of calculus,

$$\begin{aligned}\mathbb{P}_{h,\delta}(x,\xi) \geq & \mathbb{P}_h(|x|\xi,\xi) - d(x,|x|\xi) \sup_{[x,|x|\xi]} |\nabla \mathbb{P}_{h,\delta}(x,\xi)| \\ \geq & \frac{C_1}{(1-|x|^2)^{n-1}} - C_2 \frac{d(x,|x|\xi)}{(1-|x|^2)^n}.\end{aligned}$$

Then, if η is small enough to have $C_2 d(x,|x|\xi) \leq C_1(1-|x|^2)$ in $\mathcal{A}_\eta(\xi)$, then in $\mathcal{A}_\eta(\xi)$,

$$\mathbb{P}_{h,\delta}(x,\xi) \geq \frac{C}{(1-|x|^2)^{n-1}}.$$

□

4.3. **Characterization by area integral.** In this chapter we characterize $\mathcal{H}^p$ in terms of area integrals. The proof is inspired by [8] but needs an adaptation to the fact that $\mathcal{H}_\delta$-harmonic functions are not $\mathcal{H}$-harmonic. More precisely, we will prove the following part of theorem A :

Theorem 19. *For $0 < p < 2$ and u $\mathcal{H}$-harmonic, the following are equivalent :*

1. $\mathcal{M}_\alpha[u] \in L^p$ *for some* α, $0 < \alpha < 1$,
2. $S_\alpha[u] \in L^p$ *for some* α, $0 < \alpha < 1$,
3. $S_\alpha^N[u] \in L^p$ *for some* α, $0 < \alpha < 1$.

Proof. Let us show that $\|S_\beta[u]\|_p \leq C\|\mathcal{M}_\alpha[u]\|_p$. According to proposition 16, we may assume $\alpha < \beta$. Assume first that u is the Poisson integral of an L^2 function.

Notation : For a measurable function $f : \mathbb{S}^{n-1} \mapsto \mathbb{R}$, we will write

$$\lambda_f(x) = \sigma\big[\{\xi \in \mathbb{S}^{n-1} \ : \ |f(\xi)| > x\}\big].$$

Fix momentarily $\mu > 0$. Let $E = \{\xi \in \mathbb{S}^{n-1} \ : \ \mathcal{M}_\alpha[u] \leq \mu\}$ and $B = \mathbb{S}^{n-1} \setminus E$ so that $\lambda_{\mathcal{M}_\alpha[u]}(\mu) = \sigma(B)$. Let $\mathcal{R} = \mathcal{R}_\beta(E) = \bigcup_{\xi \in E} \mathcal{A}_\beta(\xi)$. There exists an increasing sequence of domains $\mathcal{R}_\varepsilon$ with $\mathcal{C}^1$ smooth boundary such that $\mathcal{R}_\varepsilon \to \mathcal{R}$.

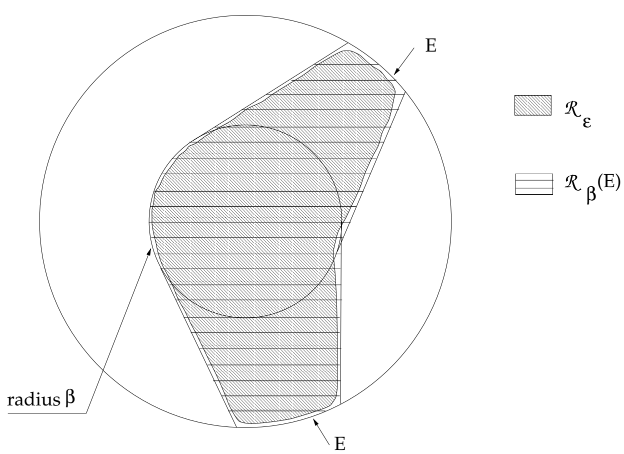

FIGURE 4. $\mathcal{R}_\beta(F)$ et $\mathcal{R}_\varepsilon$

Further, by definition of E, we have $|u(x)| \leq \mu$ in $\bigcup_{\xi \in E} \mathcal{A}_\alpha(\xi)$ thus, with lemma 15, $\left|(1-|x|^2)\nabla u\right| \leq C\mu$ in $\mathcal{R}$. Then

$$\begin{aligned}\int_E S_\beta[u]^2(\xi)d\sigma(\xi) =& \int_E \left[\int_{\mathcal{A}_\beta(\xi)} |\nabla u(x)|^2(1-|x|^2)^{-n+2}dx\right]d\sigma(\xi)\\ \leq& C\int_{\mathcal{R}} |\nabla u(x)|^2(1-|x|^2)dx\end{aligned}$$

according to lemma 10. But $L|u|^2 = 2(1-|x|^2)|\nabla u(x)|^2$, therefore

$$\begin{aligned}\int_E S_\beta[u]^2(\xi)d\sigma(\xi) \leq& C\int_{\mathcal{R}} L|u|^2(1-|x|^2)^{n-1}\frac{dx}{(1-|x|^2)^{n-1}}\\ \leq& C\overline{\lim_{\varepsilon\to 0}}\int_{\mathcal{R}_\varepsilon} L|u|^2(1-|x|^2)^{n-1}\frac{dx}{(1-|x|^2)^{n-1}} \qquad (4.1)\end{aligned}$$

Since $L(1-|x|^2)^{n-1} = -2(n-1)(1-|x|^2)^{n-1}$,

$$\begin{aligned}\int_{\mathcal{R}_\varepsilon} L|u|^2(1-|x|^2)^{n-1}\frac{dx}{(1-|x|^2)^{n-1}} =& \int_{\mathcal{R}_\varepsilon}\left[(1-|x|^2)^{n-1}L|u|^2 - |u|^2L(1-|x|^2)^{n-1}\right]\frac{dx}{(1-|x|^2)^{n-1}}\\ &-2(n-1)\int_{\mathcal{R}_\varepsilon}|u|^2dx\\ \leq& \int_{\mathcal{R}_\varepsilon}\left[(1-|x|^2)^{n-1}L|u|^2 - |u|^2L(1-|x|^2)^{n-1}\right]\frac{dx}{(1-|x|^2)^{n-1}}\end{aligned}$$

and Green's formula leads to

$$\int_E S_\beta[u]^2(\xi)d\sigma(\xi) \leq C\overline{\lim_{\varepsilon\to 0}}\int_{\partial\mathcal{R}_\varepsilon}\left[(1-|x|^2)^{n-1}\frac{\partial|u|^2}{\partial n} - |u|^2\frac{\partial(1-|x|^2)^{n-1}}{\partial n}\right]\frac{d\sigma_{\mathcal{R}_\varepsilon}(x)}{(1-|x|^2)^{n-2}}.$$

Now cut $\partial\mathcal{R}_\varepsilon$ into two parts $\partial\mathcal{R}_\varepsilon^E$ and $\partial\mathcal{R}_\varepsilon^B$ where

$$\partial\mathcal{R}_\varepsilon^E = \{r\xi\in\partial\mathcal{R}_\varepsilon\ :\ \xi\in E\}\quad\text{and}\quad \partial\mathcal{R}_\varepsilon^B = \{r\xi\in\partial\mathcal{R}_\varepsilon\ :\ \xi\in B\}.$$

— On $\partial\mathcal{R}_\varepsilon^E$, we have $d\sigma_{\mathcal{R}_\varepsilon}\sim d\sigma$. Further $\sup_{r>0}|u(r\xi)|$ and $\sup_{r>0}(1-r^2)|\nabla u(r\xi)|$ are in $L^2(\mathbb{S}^{n-1})$. Finaly, as u is the Poisson integral of an L^2 function, $\lim_{r\to 1}(1-r^2)|\nabla u(r\xi)| = 0$ almost everywhere thus, by Lebesgues' lemma,

$$\int_{\partial\mathcal{R}_\varepsilon^E}(1-|x|^2)\frac{\partial|u|^2}{\partial n}d\sigma_{\mathcal{R}_\varepsilon}(x)\to 0$$

when $\varepsilon\to 0$.

— We have already seen that $\left|(1-|x|^2)\nabla u(x)\right|\leq C\mu$ and that $|u(x)|\leq\mu$ in $\mathcal{R}$, thus

$$\int_{\partial\mathcal{R}_\varepsilon^B}(1-|x|^2)\frac{\partial|u|^2}{\partial n}d\sigma_{\mathcal{R}_\varepsilon}(x)\leq C\mu^2\int_{\partial\mathcal{R}_\varepsilon^B}d\sigma_{\mathcal{R}_\varepsilon}\leq C\mu^2\sigma(B) = C\mu^2\lambda_{\mathcal{M}_\alpha[u]}(\mu).$$

— On the other hand

$$\left|\int_{\partial\mathcal{R}_\varepsilon^E}|u|^2\frac{\partial(1-|x|^2)^{n-1}}{\partial n}\frac{d\sigma_{\mathcal{R}_\varepsilon}(x)}{(1-|x|^2)^{n-2}}\right|\leq C\int_E\mathcal{M}_\alpha[u]^2(\xi)d\sigma(\xi)\leq C\int_0^\mu t\lambda_{\mathcal{M}_\alpha[u]}(t)dt$$

since $\mathcal{M}_\alpha[u]\leq\mu$ on E.

— Finally

$$\left|\int_{\partial\mathcal{R}_\varepsilon^B}|u|^2\frac{\partial(1-|x|^2)^{n-1}}{\partial n}\frac{d\sigma_{\mathcal{R}_\varepsilon}(x)}{(1-|x|^2)^{n-2}}\right|\leq C\mu^2\int_B d\sigma = C\mu^2\sigma(B) = C\mu^2\lambda_{\mathcal{M}_\alpha[u]}(\mu).$$

But then

$$\begin{aligned}\lambda_{S_\beta[u]}(\mu) =&\sigma[\{\xi : |S_\beta(u)(\xi)| > \mu\}]\\ =&\sigma[\{\xi : |S_\beta(u)(\xi)| > \mu, \text{ and } \mathcal{M}_\alpha[u](\xi) \leq \mu\}] + \sigma[\{\xi : |S_\beta(u)(\xi)| > \mu, \text{ and } \mathcal{M}_\alpha[u](\xi) > \mu\}]\\ \leq&\frac{1}{\mu^2}\int_{\mathcal{M}_\alpha[u]\leq\mu} S_\beta[u]^2(\xi)d\sigma(\xi) + \sigma[\{\xi \ : \ \mathcal{M}_\alpha[u](\xi) > \mu\}]\\ \leq&\frac{1}{\mu^2}\int_E S_\beta[u]^2(\xi)d\sigma(\xi) + \lambda_{\mathcal{M}_\alpha[u]}(\mu)\end{aligned}$$

and taking into account the previous estimates, we get

$$\lambda_{S_\beta[u]}(\mu) \leq C\left[\lambda_{\mathcal{M}_\alpha[u]}(\mu) + \frac{1}{\mu^2}\int_0^\mu t\lambda_{\mathcal{M}_\alpha[u]}(t)dt\right]. \tag{4.2}$$

After integrating, we get

$$\begin{aligned}\|S_\beta[u]\|_p^p = p\int_0^\infty \mu^{p-1}\lambda_{S_\beta[u]}(\mu)d\mu \leq& C\int_0^\infty \mu^{p-1}\lambda_{\mathcal{M}_\alpha[u]}(\mu)d\mu + C\int_0^\infty \mu^{p-2}\int_0^\mu t\lambda_{\mathcal{M}_\alpha[u]}(t)dtd\mu\\ \leq& C\|\mathcal{M}_\alpha(u)\|_p^p + C\int_0^\infty t\lambda_{\mathcal{M}_\alpha[u]}(t)\int_t^\infty \mu^{p-3}d\mu dt\\ \leq& C\|\mathcal{M}_\alpha(u)\|_p^p\end{aligned}$$

since $0 < p < 2$.

We have shown that $\|S_\beta[u]\|_{L^p} \leq C\|u\|_{\mathcal{H}^p}$ for every $u \in \mathcal{H}^p \cap \mathbb{P}_h\big[L^2(\mathbb{S}^{n-1})\big]$. But, with help of the atomic decomposition of $\mathcal{H}^p$ (*see* [12]), $\mathbb{P}_h\big[L^2(\mathbb{S}^{n-1})\big]$ is dense in $\mathcal{H}^p$, we deduce the inequality for every $u \in \mathcal{H}^p$. $\diamond$

Let us now show the implication "(2) $\Rightarrow$ (1)" for $0 < p < 2$. More precisely, we will show that there exists a constant C such that for every $\mathcal{H}$-harmonic function u, $\|\mathcal{M}_\alpha[u]\|_p \leq C\|S_\beta[u]\|_p$.

With proposition 16, up to a change of the constant C, we may assume that $\alpha < \beta$ and $\alpha < \eta$ where η is given by proposition 18 to be such that $\mathbb{P}_{h,\delta}(x,\xi) \geq \frac{C}{(1-|x|^2)^{n-1}}$ on $\mathcal{A}_\eta(\xi)$, an estimate we will use at the end of the proof of the theorem (see the proof of the claim). Let $\frac{1}{2} < \delta < 1$. Let u be an $\mathcal{H}$-harmonic function satisfying $S_\beta[u] \in L^p(\mathbb{S}^{n-1})$, and let $u_\delta(x) = u(\delta x)$, in particular u_δ is a $\mathcal{C}^\infty$ function on $\overline{\mathbb{B}_n}$.

We will show that $\|\mathcal{M}_\alpha[u_\delta]\|_p \leq C\|S_\beta[u_\delta]\|_p$ with C a constant independent from δ. The result follows by letting $\delta \to 1$.

For $\mu > 0$, let $E = \{\xi \in \mathbb{S}^{n-1} \ : \ S_\beta[u_\delta](\xi) \leq \mu\}$ and let $B = \mathbb{S}^{n-1} \setminus E$, therefore $\lambda_{S_\beta[u_\delta]}(\mu) = \sigma(B)$. Let E_0 be the set of γ-density points of E where γ is chosen so as to be able to apply lemma 11. Let $B_0 = \mathbb{S}^{n-1} \setminus E_0$. Note that by Hardy-Littlewood's maximal theorem, $\sigma(B_0) \leq C\sigma(B) \leq C\lambda_{S_\beta[u]}(\mu)$.

Let $\mathcal{R} = \bigcup_{\xi\in E_0} \mathcal{A}_\alpha(\xi)$ and let $\mathcal{R}_\varepsilon$ be a sequence of domains with $\mathcal{C}^1$ boundary approximating $\mathcal{R}$ and such that $dist(\mathcal{R}_\varepsilon, \mathbb{S}^{n-1}) \geq \varepsilon$. We have

$$\begin{aligned}\int_{E_0} S_\beta[u_\delta](\xi)^2 d\sigma(\xi) =& \int_{E_0}\left[\int_{\mathcal{A}_\beta(\xi)} |\nabla u_\delta(x)|^2(1-|x|^2)^{-n+2}dx\right]d\sigma(\xi)\\ \geq& C\int_{\mathcal{R}}(1-|x|^2)|\nabla u_\delta(x)|^2dx\\ \geq& C\int_{\mathcal{R}_\varepsilon}(1-|x|^2)|\nabla u_\delta(x)|^2dx\end{aligned}$$

according to lemma 11.

Write $I_\delta=\displaystyle\int_{\mathcal{R}_\varepsilon}(1-|x|^2)|\nabla u_\delta(x)|^2dx$. As u_δ is $\mathcal{H}_\delta$-harmonic, $L_\delta|u_\delta|^2(x)=2(1-\delta^2|x|^2)|\nabla u_\delta(x)|^2$. Write $v(x)=(1-|x|^2)(1-\delta^2|x|^2)^{n-2}$, so that

$$\begin{aligned}I_\delta=&\int_{\mathcal{R}_\varepsilon}(1-|x|^2)(1-\delta^2|x|^2)^{n-2}L_\delta|u_\delta|^2(x)-|u_\delta|^2(x)L_\delta v(x)\frac{dx}{(1-\delta^2|x|^2)^{n-1}}\\&+\int_{\mathcal{R}_\varepsilon}|u_\delta(x)|^2\frac{L_\delta v(x)}{(1-\delta^2|x|^2)^{n-1}}dx.\end{aligned}$$

Call $\tilde{\mathcal{A}}_\beta(\xi)=\mathcal{A}_\beta(\xi)\cap\mathcal{R}_\varepsilon$, as

$$\begin{aligned}L_\delta v(x)=&-2n(1+(n-2)\delta^2-\delta^2(n-1)|x|^2)(1-\delta^2|x|^2)^{n-2}+4|x|^2(n-2)(1-\delta^2)(1-\delta^2|x|^2)^{n-3}\\\geq&-2n\bigl(1+(n-2)\delta^2\bigr)\left[1-\frac{\delta^2(n-1)}{1+(n-2)\delta^2}|x|^2\right]\geq -C\end{aligned}$$

we get

$$\begin{aligned}\int_{\mathcal{R}_\varepsilon}|u_\delta|^2\frac{L_\delta v(x)}{(1-\delta^2|x|^2)^{n-1}}dx\geq&-C\int_{E_0}\int_{\tilde{\mathcal{A}}_\beta(\xi)}|u_\delta(x)|^2\frac{1}{(1-\delta^2|x|^2)^{n-1}}dx\\\geq&-C\int_{E_0}\int_{\tilde{\mathcal{A}}_\beta(\xi)}|I_1\nabla u_\delta(x)|^2\frac{dx}{(1-\delta^2|x|^2)^{n-1}}\\\geq&-C\int_{E_0}\int_{\mathcal{A}_{\beta'}(\xi)}|\nabla u_\delta(x)|^2(1-|x|^2)^2\frac{dx}{(1-|x|^2)^{n-1}}\\\geq&-C\int_{E_0}S_{\beta'}[u_\delta](\xi)^2d\sigma(\xi)\end{aligned}$$

with lemma 13, for some $\beta'>\beta$. Finally

$$\int_{E_0}S_{\beta'}[u_\delta](\xi)^2d\sigma(\xi)\geq CJ_\delta$$

where

$$J_\delta=\int_{\mathcal{R}_\varepsilon}(1-|x|^2)(1-\delta^2|x|^2)^{n-2}L_\delta|u_\delta|^2-|u_\delta|^2L_\delta v(x)\frac{dx}{(1-\delta^2|x|^2)^{n-1}}.$$

Green's formula then leads to

$$\begin{aligned}J_\delta=&\int_{\partial\mathcal{R}_\varepsilon}(1-|x|^2)\frac{\partial|u_\delta|^2}{\partial\vec{n}}(x)-|u_\delta|^2(x)(1-|x|^2)^{2-n}\frac{\partial v(x)}{\partial\vec{n}}d\sigma_{\mathcal{R}_\varepsilon}\\\geq&C_1\int_{\partial\mathcal{R}_\varepsilon}|u_\delta|^2d\sigma_{\mathcal{R}_\varepsilon}-C_2\int_{\partial\mathcal{R}_\varepsilon}(1-|x|^2)|u_\delta||\nabla u_\delta|d\sigma_{\mathcal{R}_\varepsilon}\end{aligned}\tag{4.3}$$

since $-\frac{\partial v}{\partial\vec{n}}\geq C_1(1-r^2)^{n-2}$ and C_1,C_2 are independent from ε and from δ.

Let $K_\varepsilon=\left(\int_{\partial\mathcal{R}_\varepsilon}|u_\delta(x)|^2d\sigma_{\mathcal{R}_\varepsilon}(x)\right)^{\frac12}$ which is finite since u_δ is $\mathcal{C}^\infty$.

Then again, cut $\partial\mathcal{R}_\varepsilon$ into two parts, $\partial\mathcal{R}_\varepsilon=\partial\mathcal{R}_\varepsilon^E\cup\partial\mathcal{R}_\varepsilon^B$ with

$$\partial\mathcal{R}_\varepsilon^E=\{r\xi\in\partial\mathcal{R}_\varepsilon\ :\ \xi\in E_0\}\quad\text{and}\quad\partial\mathcal{R}_\varepsilon^B=\{r\xi\in\partial\mathcal{R}_\varepsilon\ :\ \xi\in B_0\}.$$

With lemma 15, $\left|(1-|x|^2)\nabla u_\delta(x)\right|\leq C\mu$ in $\mathcal{R}$ since $S_\beta[u_\delta](\xi)\leq\mu$ in E_0 (C independent from δ). The Cauchy-Schwartz inequality then gives

$$\begin{aligned}\int_{\partial\mathcal{R}_\varepsilon^B}(1-|x|^2)|u_\delta(x)||\nabla u_\delta(x)|d\sigma_{\mathcal{R}_\varepsilon}(x) \leq & K_\varepsilon\mu\sigma_{\mathcal{R}_\varepsilon}(\partial\mathcal{R}_\varepsilon^B)^{\frac12}\leq CK_\varepsilon\mu\sigma(B)^{\frac12}\\ \leq & CK_\varepsilon\big(\mu^2\lambda_{S_\beta[u_\delta]}(\mu)\big)^{\frac12}.\end{aligned}$$

As u_δ is $\mathcal{C}^\infty$,

$$M_\varepsilon=\int_{\partial\mathcal{R}_\varepsilon^E}(1-|x|^2)|u_\delta||\nabla u_\delta|d\sigma_{\mathcal{R}_\varepsilon}\to 0$$

when $\varepsilon\to 0$, thus (4.3) and $J_\delta\leq C\int_{E_0}S_\beta[u_\delta](\xi)^2d\sigma(\xi)$ imply

$$K_\varepsilon^2\leq C\int_{E_0}S_\beta[u_\delta](\xi)^2d\sigma(\xi)+CK_\varepsilon\big(\mu^2\lambda_{S_\beta[u_\delta]}(\mu)\big)^{\frac12}+CM_\varepsilon$$

thus, if ε is small enough,

$$K_\varepsilon^2\leq C\left[\int_{E_0}S_\beta[u_\delta](\xi)^2d\sigma(\xi)+\mu^2\lambda_{S_\beta[u_\delta]}(\mu)\right]. \tag{4.4}$$

Next, define $f_\varepsilon(\xi)=\big|u_\delta\big(r_\varepsilon(\xi)\xi\big)\big|+\mu\chi_{B_0}(\xi)$ where $r_\varepsilon(\xi)\xi$ is a parameterization of $\partial\mathcal{R}_\varepsilon$. In vue of (4.4), f_ε is an L^2 function and

$$\left(\int_{\mathbb{S}^{n-1}}|f_\varepsilon(\xi)|^2d\sigma(\xi)\right)^{\frac12}\leq 2K_\varepsilon+2\mu\sigma(B)^{\frac12}.$$

Let $U_\varepsilon(x)$ be its $\mathcal{H}_\delta$-Poisson integral.

Claim : *$|u_\delta(x)|\leq CU_\varepsilon(x)$ in $\mathcal{R}_\varepsilon$.*

We postpone the proof of this claim to the end of the proof of the theorem.

Taking a subsequence of f_ε that converges weakly to a function $f\in L^2$, it results from (4.4) that

$$\int_{\mathbb{S}^{n-1}}|f(\xi)|^2d\sigma(\xi)\leq C\left[\int_{E_0}S_\beta[u_\delta](\xi)^2d\sigma(\xi)+\mu^2\lambda_{S_\beta[u_\delta]}(\mu)\right].$$

On the other hand, as $|u_\delta(x)|\leq U_\varepsilon(x)$ in $\mathcal{R}_\varepsilon$, going to the limit, $|u_\delta(x)|\leq U(x)$ in $\mathcal{R}$ where $U=\mathbb{P}_{h,\delta}[f]$ is the $\mathcal{H}_\delta$-Poisson integral of f, thus for $x\in E_0$, $\mathcal{M}_\alpha[u_\delta](x)\leq C\mathcal{M}_\alpha[U](x)$. So

$$\int_{E_0}\mathcal{M}_\alpha[u_\delta](\xi)^2d\sigma(\xi)\leq C\int_{E_0}\mathcal{M}_\alpha[U](\xi)^2d\sigma(\xi)\leq C\int_{\mathbb{S}^{n-1}}|f(\xi)|^2d\sigma(\xi).$$

It follows that

$$\sigma\{\xi\in E_0:\mathcal{M}_\alpha[u_\delta]\geq\mu\}\leq C\left[\lambda_{S_\beta}[u_\delta](\mu)+\frac{1}{\mu^2}\int_0^\mu t\lambda_{S_\beta[u_\delta]}(t)dt\right].$$

and as $\sigma(\mathbb{S}^{n-1}\setminus E_0)=\sigma(B)\leq C\lambda_{S_\beta[u_\delta]}(\mu)$, we get

$$\lambda_{\mathcal{M}_\alpha[u_\delta]}(\mu)\leq C\left[\lambda_{S_\beta[u_\delta]}(\mu)+\frac{1}{\mu^2}\int_0^\mu t\lambda_{S_\beta[u_\delta]}(t)dt\right]$$

and an integration similar to the one after inequality (4.2) implies that there exists a constant C such that

$$\|\mathcal{M}_\alpha[u_\delta]\|_p\leq C\|S_\beta[u_\delta]\|_p.$$

We conclude by letting δ go to 1. $\diamond$

Proof of claim : Let $f_\varepsilon(x)=|u_\delta(r_\varepsilon(\xi)\xi)|+\mu\chi_{B_0}(\xi)$ et $U_\varepsilon(x)=\mathbb{P}_{h,\delta}[f_\varepsilon](x)$.

We want to show that, for $x\in\mathcal{R}_\varepsilon$, $|u_\delta(x)|\leq CU_\varepsilon(x)$. By the maximum principle, it is enough to prove this inequality on $x\in\partial\mathcal{R}_\varepsilon=\partial\mathcal{R}_\varepsilon^E\cup\partial\mathcal{R}_\varepsilon^B$.

— On $\partial\mathcal{R}_\varepsilon^E$, the inequality is true as long as we take C big enough.

— Recall that on $\mathcal{R}_\varepsilon$, $(1-|x|)|\nabla u_\delta(x)|\leq C\mu$. Then, if $x_1,x_2\in\mathcal{R}_\varepsilon$,

$$|u_\delta(x_1)-u_\delta(x_2)|\leq |x_1-x_2|\sup_{x\in[x_1,x_2]}|\nabla u_\delta(x)|.$$

Thus, if $x_1\in\mathcal{R}_\varepsilon$ and if $x_2\in B\bigl(x_1,\frac{1}{2}(1-|x_1|^2)\bigr)$, then

$$|u_\delta(x_1)-u_\delta(x_2)|\leq C\mu. \tag{4.5}$$

Fix $x_1\in\partial\mathcal{R}_\varepsilon^B$ and let $\mathcal{S}_\varepsilon$ be the portion of $\partial\mathcal{R}_\varepsilon^B$ located in the ball $B\bigl(x_1,\frac{1}{2}(1-|x_1|^2)\bigr)$. By (4.5),

$$|u_\delta(x_1)|\leq\frac{1}{\sigma_\varepsilon(\mathcal{S}_\varepsilon)}\int_{\mathcal{S}_\varepsilon}(|u_\delta(x_2)|+C\mu)d\sigma_\varepsilon(x_2).$$

Since $d\sigma_\varepsilon\simeq d\sigma$ and since $B\bigl(x_1,\frac{1}{2}(1-|x_1|^2)\bigr)$ is of radius $\frac{1}{2}(1-|x_1|^2)$, $\sigma_\varepsilon(\mathcal{S}_\varepsilon)\simeq a(1-|x_1|^2)^{n-1}$. Therefore, by definition of f_ε,

$$|u_\delta(x_1)|\leq\frac{C}{(1-|x_1|^2)^{n-1}}\int_{E_0}f_\varepsilon(\xi)d\sigma(\xi).$$

But, the Poisson kernel $\mathbb{P}_{h,\delta}(x_1,\xi)$ is $\geq\frac{c}{(1-|x_1|^2)^{n-1}}$ in $\mathcal{A}_\alpha(\xi)$, it follows that

$$|u_\delta(x_1)|\leq C\int_{E_0}\mathbb{P}_{h,\delta}(x_1,\xi)f_\varepsilon(\xi)d\sigma(\xi),$$

and thus we have $|u_\delta|\leq C\mathbb{P}_{h,\delta}[f_\varepsilon]$ in $\mathcal{R}_\varepsilon$. $\diamond$

The equivalence "(2) $\Leftrightarrow$ (3)" results immediately from lemma 14. $\square$

Remark 1 : The theorem is valid for functions u taking their values in a Hilbert space instead of $\mathbb{R}$ or $\mathbb{C}$. The key point is that equality $L|u|^2=2(1-|x|^2)|\nabla u|^2$ is valid in Hilbert spaces.

Remark 2 : For the proof of "(1) $\Rightarrow$ (2)", we have used density in $\mathcal{H}^p$ of Poisson integrals of L^2 functions obtained with the atomic decomposition. We could also use $\mathcal{H}_\delta$-harmonic functions.

4.4. **Characterization by Littlewood-Paley's g-function.** Due to the mean value inequality for $\mathcal{H}$-harmonic functions, one immediatly gets :

Lemma 20. *For every α with $0<\alpha<1$ there exists a constant C such that for every $\mathcal{H}$-harmonic function u and every $\xi\in\mathbb{S}^{n-1}$,*

$$g[u](\xi)\leq CS_\alpha[u](\xi).$$

Proof. Simply adapt the $\mathbb{R}^{n+1}_+$ case from [16]. $\square$.

Theorem 21. *Let $0<p<2$. For every $\mathcal{H}$-harmonic function u, the following are equivalent :*

1. $g[u]\in L^p(\mathbb{S}^{n-1})$,
2. $g^N[u]\in L^p(\mathbb{S}^{n-1})$,
3. $S_\alpha[u]\in L^p(\mathbb{S}^{n-1})$ *for some α, $0<\alpha<1$ (thus for every α).*

Proof. Let $\mathbb{H}$ be the Hilbert space defined by

$$\mathbb{H}=\left\{\varphi:[0,1]\mapsto\mathbb{C}\ :\ \|\varphi\|_{\mathbb{H}}^2=\int_0^1|\varphi(s)|^2(1-s^2)ds<+\infty\right\}.$$

Let u be an $\mathcal{H}$-harmonic function such that $g[u]\in L^p(\mathbb{S}^{n-1})$. For $0<s<1$ define $U(r\zeta)=Nu(rs\zeta)$, then

$$\begin{aligned}\|U(r\zeta)\|_{\mathbb{H}} = \|s \mapsto Nu(rs\zeta)\|_{\mathbb{H}} =& \int_0^1 |Nu(rs\zeta)|^2(1-s^2)ds \\ \leq& \int_0^r |\nabla u(s\zeta)|^2 \left(1-\left(\frac{s}{r}\right)^2\right)\frac{ds}{r} \\ =& \frac{1}{r^3}\int_0^r |\nabla u(s\zeta)|^2(r^2-s^2)ds \\ \leq& Cg[u](\zeta)^2\end{aligned}$$

so

$$\mathcal{M}[U](\xi) = \sup_{0<r<1} \|U(r\zeta)\|_{\mathbb{H}} \leq Cg[u](\zeta) \in L^p(\mathbb{S}^{n-1}).$$

According to remark 1 after the proof of theorem 19,

$$\|S_\alpha[U]\|_p \leq C\|\mathcal{M}[U]\|_p \leq C'\|g[u]\|_p.$$

Write $S_\alpha[U](\zeta)$ with the parameterization $r(\xi)\xi$ of $\partial\mathcal{A}_\alpha(\zeta)$:

$$S_\alpha[U](\zeta)^2 = \int_{\mathbb{S}^{n-1}}\int_0^{r(\xi)}\int_0^1 |\nabla Nu(rs\xi)|^2(1-s^2)sds(1-r^2)^{2-n}r^{n-1}drd\sigma(\xi)$$

and, with the change of variables $t = rs$, we get, changing order of integration

$$\begin{aligned}S_\alpha[U](\zeta)^2 =& \int_{\mathbb{S}^{n-1}}\int_0^{r(\xi)}\int_t^{r(\xi)} |\nabla N(t\xi)|^2 \left(1-\left(\frac{t}{r}\right)^2\right)\frac{t}{r}(1-r^2)^{2-n}r^{n-2}drdtd\sigma \\ \geq& \int_{\mathbb{S}^{n-1}}\int_0^{r(\xi)} |\nabla Nu(t\xi)|^2 \int_t^{r(\xi)}(r-t)(1-r)^{2-n}drt^{n-3}dtd\sigma(\xi)\end{aligned}$$

But, if $1-t > 2\big(1-r(\xi)\big)$

$$\int_t^{r(\xi)}(r-t)(1-r)^{2-n}dr = \int_{1-r(\xi)}^{1-t} s^{2-n}(1-t-s)ds \geq C(1-t)^{4-n}$$

thus there exists $\beta < \alpha$ such that

$$\begin{aligned}S_\alpha[U](\zeta)^2 \geq& \int_{\mathcal{A}_\beta(\zeta)} \left|N^2u(t\xi)\right|^2(1-t)^{4-n}t^{n-1}dtd\sigma(\zeta) \\ \geq& C\int_{\mathcal{A}_{\beta'}(\zeta)} \left|I_1N^2u(t\xi)\right|^2(1-t)^{2-n}t^{n-1}dtd\sigma(\zeta)\end{aligned}$$

with $\beta' < \beta$ according to lemma 13, thus $S_\alpha[U](\zeta)^2 \geq CS^N_{\beta'}[u](\zeta)$, which completes the proof of (1) $\Leftrightarrow$ (2).

The equivalence (1) $\Leftrightarrow$ (3) results directly from lemma 14. □

5. Characterization of Hardy-Sobolev spaces

In this section, we prove theorems B and C.

In these theorems, that $\mathcal{M}$ can be replaced by $\mathcal{M}_\alpha$ is a direct consequence of the mean value inequality (*see* proposition 16 and the remarks following it). We will need the following.

Notation : For an integer $k \geq 1$, write $\mathbb{A}_k$ for the set of indices

$$\mathbb{A}_k = \{(i,j) \in \mathbb{N}\times\mathbb{N} \ : \ 0 \leq i \leq k,\ 0 \leq j \leq k+1,\ j \text{ even},\ 1 \leq i+j \leq k+1\}.$$

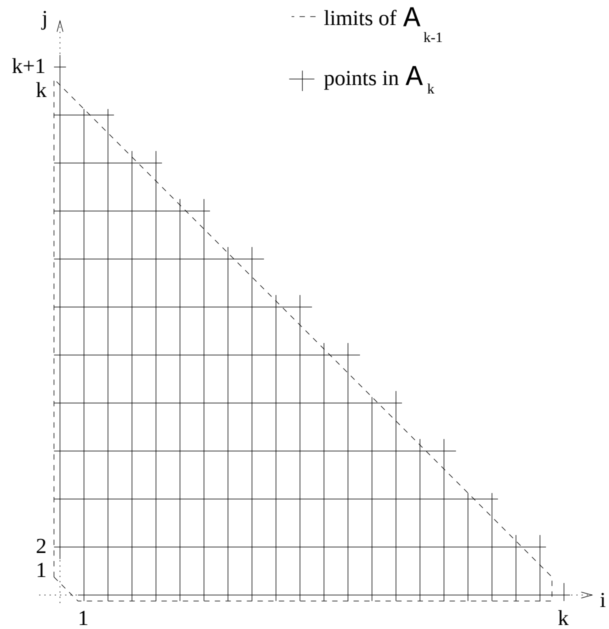

FIGURE 5. The set A_k (here with k even)

Lemma 22. *For every $k \geq 1$, there exists two families of polynomials $\big(P_j^{(k)}\big)_{j=1\ldots\left[\frac{k}{2}\right]}$ and $\big(Q_{i,j}^{(k)}\big)_{(i,j)\in\mathbb{A}_k}$ such that for every $\mathcal{H}$-harmonic function u and every k,*

$$(1-r^2)N^{k+1}u + 2(n-1-k)N^k u = \sum_{j=1}^{\left[\frac{k}{2}\right]} P_j^{(k)}(r)\Delta_\sigma^j u + (1-r^2)\sum_{(i,j)\in\mathbb{A}_k} Q_{i,j}^{(k)}(r)N^i\Delta_\sigma^{\frac{j}{2}} u.$$

Moreover, the polynomials $P_{[k/2]}^{(k)}$ and $Q_{0,\left[\frac{k+1}{2}\right]}^{(k)}$ are not zero on the boundary.

Proof. Using the radial-tangential expression of D, we can see that if $Du = 0$ then

$$(1-r^2)N^2u + 2(n-2)Nu = (1-r^2)[(n-2)Nu - \Delta_\sigma u]. \tag{5.1}$$

The lemma is thus verified for $k = 1$ with $Q_{1,0}^{(1)}(r) = n-2$, $Q_{0,2}^{(1)}(r) = -1$.

Applying N^{k-1} to equation (5.1) leads to

$$\begin{aligned}(1-r^2)N^{k+1}u + \big(n-1-k\big)N^k u =& r^2\sum_{l=1}^{k-1} a_l^{(k)}N^l u + r^2\sum_{l=0}^{k-2} b_l^{(k)}N^l\Delta_\sigma u\\ &+ (1-r^2)\big[(n-2k)N^k u - N^{k-1}\Delta_\sigma u\big]\end{aligned}$$

and we conclude with the induction hypothesis. □

The equivalence of $1a$ and $1b$ as well as the equivalence of $2a$ and $2b$ in theorem C have already been shown. We will now prove the remaining of this theorem.

Theorem 23. *For $0 < \alpha < 1$, $0 < p < +\infty$, for every integer $0 \leq k \leq n-2$ and for every $\mathcal{H}$-harmonic function u, the following are equivalent :*

1. *If k is* even
   (a) $\mathcal{M}_\alpha\big[N^j u\big] \in L^p(\mathbb{S}^{n-1})$*, for $0 \leq j \leq k$,*
   (b) $\mathcal{M}_\alpha\big[\Delta_\sigma^j u\big] \in L^p(\mathbb{S}^{n-1})$*, for $0 \leq j \leq \frac{k}{2}$,*
   (c) $\mathcal{M}_\alpha\big[\nabla^j u\big] \in L^p(\mathbb{S}^{n-1})$ *for $0 \leq j \leq k$.*

2. *If k is* odd
   (a) $\mathcal{M}_\alpha\big[N^j u\big] \in L^p(\mathbb{S}^{n-1})$, *for* $0 \le j \le k$,
   (b) $\mathcal{M}_\alpha\big[\Delta_\sigma^j u\big] \in L^p(\mathbb{S}^{n-1})$, *for* $0 \le j \le \frac{k-1}{2}$, *and* $\mathcal{M}_\alpha\Big[(1-r^2)\Delta_\sigma^{\frac{k+1}{2}} u\Big] \in L^p(\mathbb{S}^{n-1})$.

Remark : If $\mathcal{M}_\alpha\Big[\Delta_\sigma^{\frac{k}{2}} u\Big] \in L^p(\mathbb{S}^{n-1})$ then, by the mean value properties, we get that

$$\mathcal{M}_\alpha\Big[(1-r^2)\Delta_\sigma^{\frac{k+1}{2}} u\Big] \in L^p(\mathbb{S}^{n-1}).$$

Hence $\mathcal{H}^p_k \subset \mathcal{H}^p_{k,N}$.

Proof. The theorem is of course true for $k = 0$. Assume the result holds up to rank $k-1$.

Assume first that k is *even*. Implication $(c) \Rightarrow (b)$ is obvious. Let us show $(b) \Rightarrow (a)$.

Let u be an $\mathcal{H}$-harmonic function that satisfies (b), and let

$$v = 2(n-1-k)N^k u + (1-r^2)N^{k+1}u. \tag{5.2}$$

According to lemma 22,

$$v(r\zeta) = \sum_{j=1}^{\left[\frac{k}{2}\right]} P_j^{(k)}(r)\Delta_\sigma^j u + (1-r^2)\sum_{(i,j)\in\mathbb{A}_k} Q_{i,j}^{(k)}(r) N^i \Delta_\sigma^{\frac{j}{2}} u.$$

Then, with hypothesis (b), for $0 \le j \le \frac{k}{2}$, $\mathcal{M}_\alpha\big[\Delta_\sigma^j u\big] \in L^p$ thus

$$\mathcal{M}_\alpha\left[\sum_{j=1}^{\left[\frac{k}{2}\right]} P_j^{(k)}(r)\Delta_\sigma^j u\right] \in L^p(\mathbb{S}^{n-1}).$$

On the other hand, corollary 9 implies that

$$\mathcal{M}_\alpha\left[(1-r^2)\sum_{(i,j)\in\mathbb{A}_k} Q_{i,j}^{(k)}(r) N^i \Delta_\sigma^{\frac{j}{2}} u\right] \le C \sum_{(i,j)\in\mathbb{A}_{k-1}\cup(0,0)} \mathcal{M}_\beta\Big[N^i \Delta_\sigma^{\frac{j}{2}} u\Big] \in L^p(\mathbb{S}^{n-1}),$$

since by proposition 16 $\big\|\mathcal{M}_\alpha\big[N^i\varphi\big]\big\|_{L^p}$ and $\big\|\mathcal{M}_\beta\big[N^i\varphi\big]\big\|_{L^p}$ are equivalent for $\mathcal{H}$-harmonic functions and since, by induction hypothesis, $\mathcal{M}_\alpha\Big[N^i\Delta_\sigma^{\frac{j}{2}}u\Big] \in L^p(\mathbb{S}^{n-1})$. We deduce from it that $\mathcal{M}_\alpha[v] \in L^p(\mathbb{S}^{n-1})$. But, solving the differential equation (5.2), we get

$$N^k u(r\zeta) = \left(\frac{1-r^2}{r^2}\right)^{n-1-k} \int_0^r v(t\zeta)\frac{t^{2n-3-2k}}{(1-t^2)^{n-k}}dt \tag{5.3}$$

thus $\mathcal{M}_\alpha\big[N^k u\big] \in L^p(\mathbb{S}^{n-1})$ since $k \le n-2$.

Assume now that u satisfies (a) *i.e.* that $\mathcal{M}_\alpha\big[N^j u\big] \in L^p(\mathbb{S}^{n-1})$ for $j \le k$ and let us show that $\mathcal{M}_\alpha\big[\Delta_\sigma^j u\big] \in L^p(\mathbb{S}^{n-1})$ for $j \le \frac{k}{2}$. Let $1 > \beta > \alpha$.

According to the induction hypothesis, for $j \le \frac{k}{2} - 1$, $\mathcal{M}_\alpha\big[\Delta_\sigma^j u\big] \in L^p(\mathbb{S}^{n-1})$ then

$$\mathcal{M}_\alpha\left[\sum_{j=1}^{\frac{k}{2}-1} P_j^{(k)}(r)\Delta_\sigma^j u\right] \in L^p(\mathbb{S}^{n-1}).$$

With corollary 9, $\mathcal{M}_\alpha\big[(1-r^2)N^{k+1}u\big] \le C\mathcal{M}_\beta\big[N^k u\big] \in L^p(\mathbb{S}^{n-1})$, by proposition 16. As, for a regular function φ,

$$\mathcal{M}_\alpha\big[(1-r^2)\varphi\big] \le C\big(\mathcal{M}_\alpha\big[(1-r^2)^2 N\varphi\big] + |\varphi(0)|\big).$$

Finally, iterating this inequality, for $(i,j) \in \mathbb{A}_k$,

$$\begin{aligned}\mathcal{M}_\alpha\Big[(1-r^2)Q_{i,j}^{(k)}(r)N^i\Delta_\sigma^{\frac{j}{2}}u\Big] \leq & C\mathcal{M}_\alpha\Big[(1-r^2)N^i\Delta_\sigma^{\frac{j}{2}}u\Big]\\ \leq & C\mathcal{M}_\alpha\Big[(1-r^2)^{k-i+1}N^k\Delta_\sigma^{\frac{j}{2}}u\Big]+C\sum_{j=0}^{k-1}\big|\nabla^j u(0)\big|\\ \leq & C\sum_{j=0}^{k}\mathcal{M}_\beta\big[N^j u\big],\end{aligned}$$

by corollary 9 and $k-i+1\geq j$ since $(i,j)\in\mathbb{A}_k$. So, with proposition 16, $C\sum_{j=0}^k\mathcal{M}_\beta\big[N^ju\big]\in L^p(\mathbb{S}^{n-1})$, thus

$$\mathcal{M}_\alpha\left[\sum_{(i,j)\in\mathbb{A}_k}(1-r^2)Q_{i,j}^{(k)}(r)N^i\Delta_\sigma^{\frac{j}{2}}u\right]\in L^p(\mathbb{S}^{n-1}).$$

We then get from lemma 22 that $\mathcal{M}_\alpha\Big[P_{k/2}^{(k)}(r)\Delta_\sigma^{k/2}u\Big]\in L^p(\mathbb{S}^{n-1})$, and as $P_{k/2}^{(k)}$ is not zero on the boundary, $\mathcal{M}_\alpha\Big[\Delta_\sigma^{k/2}u\Big]\in L^p(\mathbb{S}^{n-1})$. So (a) and (b) are equivalent.

Let us now show that $(a)+(b)$ implies (c). It is enough to show this implication for $\mathbb{X}$ a differential operator of the form $\mathbb{X}=N^j\mathbb{Y}$ with $\mathbb{Y}$ a product of $k-j$ operators of the form $\mathcal{L}_{i,j}$. We can assume that $j<k$. Let

$$v=(1-r^2)N^{j+1}u+2(n-1-j)N^ju$$

and compose with $\mathbb{Y}$, it results that

$$\mathbb{Y}v=(1-r^2)N^{j+1}\mathbb{Y}u+2(n-1-j)N^j\mathbb{Y}u.$$

Using as previously formula (5.3), we see that

$$\mathcal{M}_\alpha[\mathbb{X}u]\in L^p(\mathbb{S}^{n-1})$$

which completes the proof in the case k is even.

Assume now that k is *odd*. The proof of $(b)\Rightarrow(a)$ is similar to case k even. The converse is again based on lemma 22. According to the induction hypothesis, $\mathcal{M}_\alpha\big[\Delta_\sigma^l u\big]\in L^p(\mathbb{S}^{n-1})$ for $0\leq l\leq\frac{k-1}{2}=\big[\frac{k}{2}\big]$ so that

$$\mathcal{M}_\alpha\left[\sum_{j=1}^{\left[\frac{k}{2}\right]}P_j^{(k)}(r)\Delta_\sigma^j u\right]\in L^p(\mathbb{S}^{n-1}).$$

One has, as before,

$$\mathcal{M}_\alpha\left[(1-r^2)\sum_{(i,j)\in\mathbb{A}_k\setminus(0,k+1)}Q_{i,j}^{(k)}(r)N^i\Delta_\sigma^{\frac{j}{2}}u\right]\in L^p(\mathbb{S}^{n-1})$$

and that $\mathcal{M}_\alpha\big[(1-r^2)N^{k+1}\big]\in L^p(\mathbb{S}^{n-1})$.

Combining all this, we get that

$$\mathcal{M}_\alpha\Big[(1-r^2)Q_{0,k+1}^{(k)}(r)\Delta_\sigma^{\frac{k+1}{2}}u\Big]\in L^p(\mathbb{S}^{n-1})$$

and as $Q_{0,k+1}^{(k)}$ is non-zero on the boundary, we finaly get

$$\mathcal{M}_\alpha\Big[(1-r^2)\Delta_\sigma^{\frac{k+1}{2}}u\Big]\in L^p(\mathbb{S}^{n-1})$$

and (a) and (b) are equivalent. □

We will now prove the area integral characterization in theorem B.

Proof of theorem B. The fact that S_α can be replaced by S_α^N is a direct consequence of lemma 14 (with $\gamma = -\frac{n}{2}+1$). Further, as

$$\left|N\Delta_\sigma^{k/2}u\right| \leq CI_k(N^{k+1}\Delta_\sigma^{k/2}u) + \sup_{0\leq j\leq 2k, |z|\leq\varepsilon}\left|\nabla^j u\right|,$$

so, lemma 14 and the mean value inequality imply that

$$S_\alpha^N\left[\Delta_\sigma^{k/2}u\right] \leq S_\beta^N\left[N^k u\right] + \|u\|_{\mathcal{H}^p},$$

so that 8 implies 5.

Let us now prove that if, for $0\leq j\leq\frac{k}{2}$, $S_\alpha\left[\Delta_\sigma^j u\right]\in L^p(\mathbb{S}^{n-1})$, then $S_\alpha^N\left[N^k u\right]\in L^p(\mathbb{S}^{n-1})$. The proof goes according to the method developped for the equivalence of maximal functions.

For simplicity, we will restrict our attention to the case $k=1$. In order to estimate $S_\alpha^N[Nu]$, we have to estimate Nu. While trying to use the previous method, lemma 22 for $k=2$ does not give a satisfying estimate. However, we can obtain the desired estimate as follows. Denote by v the function

$$v = 2(n-2)Nu + (1-r^2)N^2u,$$

then $Nv = 2(n-3)N^2u+(1-r^2)N^3u+2(1-r^2)N^2u$ and write this in the form $Nv = w+2(1-r^2)N^2u$. As before, solving the differential equation $(1-r^2)N^3u+2(n-3)N^2u = w$, we have

$$N^2u(r\zeta) = \left(\frac{1-r^2}{r^2}\right)^{n-3}\int_0^r w(tz)t^{2(n-3)+1}(1-t^2)^{2-n}dt$$

so that

$$\left|N^2u(r\zeta)\right| \leq C(1-r^2)^{n-3}I_{3-n}(|w|). \tag{5.4}$$

On the other hand, by lemma 22 for $k=1$,

$$v = (n-2)(1-r^2)Nu - (1-r^2)\Delta_\sigma u,$$

so

$$Nv = (n-2)(1-r^2)N^2u - (1-r^2)N\Delta_\sigma u + 2r^2\Delta_\sigma u - 2r^2(n-2)Nu.$$

Recall that $|f|\leq CI_l(\left|N^l f\right|)+C\sup_{|z|<\varepsilon, j\leq l}\left|\nabla^j f\right|$, and that $I_{k+1}(|f|)\leq I_k(|f|)$. Using these facts, one gets

$$\begin{aligned}|w| \leq& |Nv| + 2(1-r^2)\left|N^2u\right| \\ \leq& C\left(I_1\left(|N\Delta_\sigma u|\right) + I_1\left(\left|N^2u\right|\right) + (1-r^2)\left|N^2u\right| + (1-r^2)|N\Delta_\sigma u|\right) + \sup_{0\leq j\leq 3, |z|\leq\varepsilon}\left|\nabla^j u\right|.\end{aligned}$$

Inserting this in (5.4), and invoking the facts that $I_l\big((1-r^2)|f|\big) = I_{l+1}(|f|)$ and that $I_l(I_s|f|)\leq CI_{l+s}(|f|)$, one gets

$$\left|N^2u(r\zeta)\right| \leq C(1-r^2)^{n-3}\left(I_{4-n}(|N\Delta_\sigma u|) + I_{4-n}(\left|N^2u\right|) + \sup_{|z|<\varepsilon, 0\leq j\leq 3}\left|\nabla^j u\right|\right).$$

We are now in position to estimate $S_\alpha^N[Nu]$:

$$\begin{aligned}S_\alpha^N[Nu](\zeta)^2=&\int_{\mathcal{A}_\alpha(\zeta)}\left|N^2u(x)\right|^2(1-|x|)^{2-n}dx\\ \leq& C\left(\int_{\mathcal{A}_\alpha(\zeta)}[I_{4-n}(|N\Delta_\sigma u|)]^2(1-|x|^2)^{n-4}dx\right.\\ &\left.+\int_{\mathcal{A}_\alpha(\zeta)}\left[I_{4-n}(\left|N^2u\right|)\right]^2(1-|x|^2)^{n-4}dx+\sup_{|z|<\varepsilon,0\leq j\leq 3}\left|\nabla^j u\right|^2\right)\end{aligned}$$

A further appeal to lemma 13, with $l=4-n$, $d=0$, $k=2$ and $\gamma=\frac{n-4}{2}$ leads to

$$\int_{\mathcal{A}_\alpha(\zeta)}\left[I_{4-n}(\left|N^2u\right|)\right]^2(1-|x|^2)^{n-4}dx\leq C\int_{\mathcal{A}_\beta(\zeta)}\left|N^2u\right|^2(1-|x|^2)^{4-n}dx.$$

A last appeal to lemma 13, with $l=4-n$, $d=0$, $k=1$ and $\gamma=\frac{n-4}{2}$ leads to

$$\int_{\mathcal{A}_\alpha(\zeta)}[I_{4-n}(|N\Delta_\sigma u|)]^2(1-|x|^2)^{n-4}dx\leq C\int_{\mathcal{A}_\beta(\zeta)}|N\Delta_\sigma u|^2(1-|x|^2)^{4-n}dx\leq CS_\gamma^N\left[\Delta_\sigma^{\frac12}u\right],$$

by the mean value properties. As the only part that matters in this last integral is the part near to the boundary, we will cut it into two parts. Let κ be a constant that we will fix later. Then

$$\begin{aligned}\int_{\mathcal{A}_\beta(\zeta)}\left|N^2u\right|^2(1-|x|^2)^{4-n}dx\leq&\int_{\mathcal{A}_\beta(\zeta)\cap B(0,\kappa)}\left|N^2u\right|^2(1-|x|^2)^{4-n}dx\\ &+\int_{\mathcal{A}_\beta(\zeta)\cap\left(\mathbb{B}_n\setminus B(0,\kappa)\right)}\left|N^2u\right|^2(1-|x|^2)^{4-n}dx\\ \leq& C\sup_{|z|\leq\kappa,0\leq j\leq 3}\left|\nabla^j u\right|+(1-\kappa^2)^2\int_{\mathcal{A}_\beta(\zeta)}\left|N^2u\right|^2(1-|x|^2)^{2-n}dx.\end{aligned}$$

Grouping the above estimates, we finaly get

$$S_\alpha^N[Nu](\zeta)^2\leq CS_\gamma^N\left[\Delta_\sigma^{\frac12}u\right](\zeta)^2+C(1-\kappa^2)^2S_\beta[Nu](\zeta)^2+C\sup_{|z|\leq\kappa,0\leq j\leq 3}\left|\nabla^j u\right|. \tag{5.5}$$

But this inequality depends only on the mean value inequality, in particular, one can replace u in (5.5) by $u_\delta(x)=u(\delta x)$ and get

$$S_\alpha^N[Nu_\delta](\zeta)^2\leq CS_\gamma^N\left[\Delta_\sigma^{\frac12}u_\delta\right](\zeta)^2+C(1-\kappa^2)^2S_\beta^N[Nu_\delta](\zeta)^2+C\sup_{|z|\leq\kappa,0\leq j\leq 3}\left|\nabla^j u_\delta\right|$$

with constants independant on $\frac12<\delta<1$. Then taking $L^p(\mathbb{S}^{n-1})$ norms, one gets

$$\begin{aligned}\left\|S_\alpha^N[Nu_\delta]\right\|_p\leq& C\left\|S_\gamma^N\left[\Delta_\sigma^{\frac12}u_\delta\right]\right\|_p+C(1-\kappa^2)\left\|S_\beta^N[Nu_\delta]\right\|_p+C\|u\|_{\mathcal{H}^p}\\ \leq& C\left\|S_\gamma^N\left[\Delta_\sigma^{\frac12}u\right]\right\|_p+C'(1-\kappa^2)\left\|S_\alpha^N[Nu_\delta]\right\|_p+C\|u\|_{\mathcal{H}^p},\end{aligned}$$

with lemma 12. It is now enough to choose κ such that $C'(1-\kappa^2)=\frac12$, then

$$\left\|S_\alpha^N[Nu_\delta]\right\|_p\leq\frac{C}{2}\left\|S_\gamma^N\left[\Delta_\sigma^{\frac12}u\right]\right\|_p+\frac{C}{2}\|u\|_{\mathcal{H}^p}.$$

We then conclude by monotone convergence when $\delta\to 1$.

So far we have proved the equivalence of properties 1 to 8 and that 9 implies these properties. To see that 9 is actually equivalent to them, it is enough to see that in 9, ∇^k can be replaced by any operator of the form $N\mathbb{Y}$ where $\mathbb{Y}$ is a product of $k-1$ operators of the form $\mathcal{L}_{i,j}$. This is then a direct consequence of 2 and 7. $\square$

6. Lipschitz spaces and Zygmund classes

6.1. Lipschitz spaces.

Lemma 24. *Let $k \in \mathbb{N}$ and $0 < \alpha < 1$. Assume further that if n is odd then $k \leq n-2$. There exists a constant C such that for every $f \in \mathcal{C}^{k+\alpha}(\mathbb{S}^{n-1})$, for every $r\xi \in \mathbb{B}_n$,*

$$(1-r^2)^k \left|\nabla^k \mathbb{P}_h[f](r\xi)\right| \leq C(1-r^2)^\alpha.$$

In particular, $\mathbb{P}_h[f] \in \mathcal{C}^{k+\alpha}(\overline{\mathbb{B}_n})$.

Proof. Fix $\xi_0 \in \mathbb{S}^{n-1}$, there exists $P_{\xi_0}^{(k)}$, a combination of spherical harmonics of order less than k such that the Taylor polynomials of order k at ξ_0 of $P_{\xi_0}^{(k)}$ and of f coincide. Then

$$\left|f(\xi) - P_{\xi_0}^{(k)}(\xi)\right| \leq C(1- <\xi, \xi_0>)^\alpha.$$

But then, $\mathbb{P}_h[f] = \mathbb{P}_h\left[P_{\xi_0}^{(k)}\right] + \mathbb{P}_h\left[f - P_{\xi_0}^{(k)}\right]$.

From the spherical harmonics expansion of $\mathbb{P}_h$, if n is *odd* and $k \leq n-2$, or if n is *even*, there exists a constant C, independent of ξ_0 such that

$$\left\|\mathbb{P}_h\left[P_{\xi_0}^{(k)}\right]\right\|_{\mathcal{C}^k} \leq C\|f\|_{\mathcal{C}^k}.$$

To estimate $\nabla^k \mathbb{P}_h\left[f - P_{\xi_0}^{(k)}\right]$ we need the following estimates on the hyperbolic Poisson kernel :

1. $\left|\nabla^k \mathbb{P}_h(r\zeta, \xi)\right| \leq \dfrac{C}{(1-r)^{n-1+k}}$,
2. $\left|\nabla^k \mathbb{P}_h(r\zeta, \xi)\right| \leq \dfrac{C}{(1- <\zeta, \xi>)^{n-1+k}}$ provided $r\zeta \in \mathcal{A}_{\alpha_k}(\xi)$ for some α_k small enough.

Both estimates result directly from the mean value inequalities applied to the $\mathcal{H}$-harmonic function $u(r\zeta) = \mathbb{P}_h(r\zeta, \xi)$, $\xi \in \mathbb{S}^{n-1}$ fixed.

Furthermore, we are only interested in the estimates when $r\zeta$ is "near" to $\mathbb{S}^{n-1}$, *i.e.* r near to 1. In this case (2) holds when $1 - \langle \zeta, \xi \rangle < c_k(1-r)$. Then

$$\begin{aligned}
(1-r)^k \left|\nabla^k \mathbb{P}_h\left[f - P_{\xi_0}^{(k)}\right](r\xi_0)\right| \leq & (1-r)^k \int_{\mathbb{S}^{n-1}} \left|\nabla^k \mathbb{P}_h(r\zeta, \xi_0)\right| \left|f(\xi) - P_{\xi_0}^{(k)}\right| d\sigma(\xi) \\
\leq & (1-r)^k \int_{1-\langle \xi, \xi_0 \rangle < c_k(1-r)} \left|\nabla^k \mathbb{P}_h(r\zeta, \xi_0)\right| \left|f(\xi) - P_{\xi_0}^{(k)}\right| d\sigma(\xi) \\
& + (1-r)^k \int_{1-\langle \xi, \xi_0 \rangle > c_k(1-r)} \left|\nabla^k \mathbb{P}_h(r\zeta, \xi_0)\right| \left|f(\xi) - P_{\xi_0}^{(k)}\right| d\sigma(\xi) \\
\leq & \frac{C}{(1-r)^{n-1}} \int_{1-\langle \xi, \xi_0 \rangle < c_k(1-r)} (1- <\xi, \xi_0>)^\alpha d\sigma(\xi) \\
& + C \int_{1-\langle \xi, \xi_0 \rangle > c_k(1-r)} (1- <\xi, \xi_0>)^{-(n-1)+\alpha} d\sigma(\xi)
\end{aligned}$$

where for the first integral we have used estimate (1) on $\mathbb{P}_h$ and for the second we have used estimate (2). This immediatly leads to the desired result. □

Remark : When n is even, lemma 3 and the result in the Euclidean case give directly the result.

The converse of this result can also be obtained by a transfer from the Euclidean case :

Lemma 25. *Let $k \in \mathbb{N}$ and $0 < \alpha < 1$. Let f be a distribution on $\mathbb{S}^{n-1}$ and let $u = \mathbb{P}_h[f]$. Assume that there exists a constant C such that u satisfies the following inequality :*

$$(1-r^2)^k \left|\nabla^k u(r\xi)\right| \leq C(1-r^2)^\alpha.$$

Then $f \in \mathcal{C}^{k+\alpha}(\mathbb{S}^{n-1})$.

Proof. Let $v = \mathbb{P}_e[f]$. Then, by lemma 4, for $r\zeta \in \mathbb{B}_n$,

$$v(r\zeta) = \int_0^1 \eta(r,s)u(rs\zeta)ds.$$

But the estimates on η imply that

$$(1-r^2)^k\left|\nabla^k v(r\xi)\right| \leq C(1-r^2)^\alpha.$$

Thus, with the result on Euclidean harmonic functions, $f \in \mathcal{C}^{k+\alpha}(\mathbb{S}^{n-1})$. □

Remark : When n is odd, if $k \geq n-1$, the condition $(1-r^2)^k\left|\nabla^k u(r\xi)\right| \leq C(1-r^2)^\alpha$. is reduced to u constant.

6.2. **Zygmund classes.** Let us fix $\xi_0 \in \mathbb{S}^{n-1}$ and denote for $\xi \in \mathbb{S}^{n-1}$ by R_ξ a rotation on $\mathbb{S}^{n-1}$ that maps ξ_0 to ξ. Let R_ξ^* be the reverse rotation that maps ξ to ξ_0.

Define the Zygmund class of order n on $\mathbb{S}^{n-1}$ by

$$\mathbb{Z}_n(\mathbb{S}^{n-1}) = \{f \in \mathcal{C}^{n-2}(\mathbb{S}^{n-1}) \ : \ \left|\tilde{\Delta}_\xi^n f(\zeta)\right| \leq C(1-\langle\zeta,\xi\rangle)^{n-1}\}$$

where $\tilde{\Delta}_\xi^n$ is the difference operator defined by induction on j by

$$\tilde{\Delta}_\xi^1 f(\zeta) = f(R_\xi\zeta) - f(\zeta)$$

and $\tilde{\Delta}_\xi^{j+1} f = \tilde{\Delta}_\xi^1(\tilde{\Delta}_\xi^j f)$.

Define the Zygmund class of order n on $\mathbb{B}_n$ by

$$Z_n(\mathbb{B}_n) = \{f \in \mathcal{C}^{n-2}(\mathbb{B}_n) \ : \ \|\Delta_h^{n,\gamma} f(\zeta)\|_{L^\infty(\mathbb{B}_n)} \leq C|h|^{n-1} \text{ for any curve } \gamma : [0,1] \mapsto \mathbb{B}_n\}$$

where Δ_h^j is the difference operator along γ also defined inductively by

$$\Delta_h^{1,\gamma} u(z) = u \circ \gamma(h) - u \circ \gamma(0)$$

where $\gamma(0) = z$, and $\Delta_h^{j+1,\gamma} u = \Delta_h^{1,\gamma}(\Delta_h^{j,\gamma} u)$.

It follows from standard methods (using the mean-value properties) that the set of $\mathcal{H}$-harmonic functions belonging to the Zygmund class is given by :

$$\begin{aligned}\{u\ \mathcal{H}-\text{harmonic},\ u \in Z_n(\mathbb{B}_n)\} =& \left\{u \in \mathcal{C}^{n-1}(\mathbb{B}_n),\ u\ \mathcal{H}-\text{harmonic},\ |\nabla^n u(z)| \leq \frac{C}{1-|z|}\right\} \\ =& \left\{u \in \mathcal{C}^{n-1}(\mathbb{B}_n),\ u\ \mathcal{H}-\text{harmonic},\ |N^n u(z)| \leq \frac{C}{1-|z|}\right\}.\end{aligned}$$

The next theorem states that this class is the set of $\mathcal{H}$-harmonic extensions of members of the Zygmund class of order n on $\mathbb{S}^{n-1}$.

Theorem 26. *A function f belongs to $\mathbb{Z}_n(\mathbb{S}^{n-1})$ if and only if $u = \mathbb{P}_h[f]$ belongs to $\mathbb{Z}_n(\mathbb{B}_n)$.*

Proof. The proof follows from standard arguments. Let us first prove that $\mathbb{P}_h[f] \in \mathbb{Z}_n(\mathbb{B}_n)$ when $f \in \mathbb{Z}_n(\mathbb{S}^{n-1})$. Note that, for fixed $\xi \in \mathbb{S}^{n-1}$,

$$\int_{\mathbb{S}^{n-1}} N_z^j \mathbb{P}_h(z\zeta,\xi)d\sigma(\zeta) = 0$$

for any $j \geq 1$, since $\mathbb{P}_h$ has integral 1 on $\mathbb{S}^{n-1}$. Using this fact and the symmetry under rotations of $\mathbb{P}_h(r\zeta,.)$, we get that

$$\int_{S^{n-1}} \mathbb{P}_h(r\zeta,\xi)f(\xi)d\sigma(\xi) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} N^n\mathbb{P}_h(r\zeta,\xi)\tilde{\Delta}_\xi^n f\big((R_\xi^*)^{n-1}\xi_0\big)d\sigma(\xi).$$

Now, by assumption, $\left|\tilde{\Delta}_\xi^n f\big((R_\xi^*)^{n-1}\xi_0\big)\right| \leq C(1-\langle\xi,\xi_0\rangle)^{n-1}$, so that the desired estimate follows as in the proof of lemma 24.

For the converse, by the same proof as in lemma 25, we get that $v = \mathbb{P}_e[f]$ belongs to $\mathbb{Z}_n(\mathbb{B}_n)$ and we conclude that $f \in \mathbb{Z}_n(\mathbb{S}^{n-1})$ from the euclidean harmonic theory.
Remark : It is proved in [12] that any $\mathcal{H}$-harmonic function u is at most in $\mathbb{Z}_n(\mathbb{B}_n)$. In other words, it means that the Zygmund class of order n is the limit class preserved by the hyperbolic Poisson kernel.

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